# L3 [M 4.3, 4.4] LINEAR INDEPENDENCE, BASIS + DIMENSION

**DEF 1** A V.S. $V$ is FINITE DIMENSIONAL if it has a finite spanning set, ie $\exists\ \vec{v}_1, \dots, \vec{v}_n$ :

$$V = \mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_n\}.$$

**EX 2** $\mathbb{R}^n$ is F.D. with $\mathbb{R}^n = \mathrm{SPAN}\{\vec{e}_1, \dots, \vec{e}_n\}$

as if $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$ then $\vec{v} = \sum_{j=1}^{n} v_j \vec{e}_j$

**EX 3** Let $V = \mathrm{SPAN}\left\{ \underset{\vec{v}_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{\vec{v}_3}{\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}} \right\} \subseteq \mathbb{R}^3.$

Although $\mathcal{S} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $V$ it contains a redundancy as

$$\vec{v}_3 = 2\vec{v}_1 + 3\vec{v}_2 \quad (*)$$

**CLAIM** $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathrm{Span}\{\vec{v}_1, \vec{v}_2\} = \text{PLANE } z=0.$

**PF** $\boxed{\supseteq}$ EASY

$\boxed{\subseteq}$ Let $\vec{w} \in \mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

So

$$\begin{aligned} \vec{w} &= \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 \\ &= \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 (2\vec{v}_1 + 3\vec{v}_2) \quad (*) \\ &= (\alpha_1 + 2\alpha_3)\vec{v}_1 + (\alpha_2 + 3\alpha_3)\vec{v}_2 \in \mathrm{Span}\{\vec{v}_1, \vec{v}_2\} \end{aligned}$$

D

**GOAL** Given a finite spanning set $\mathcal{S}$ for a FDVS $V$ eliminate any redundancies to obtain a MINIMAL SPANNING SET.

**DEF 4** A finite subset $\mathcal{S}$ of $V$ is a MINIMAL SPANNING SET for $V$ if

(a) $V = \mathrm{Span}(\mathcal{S})$

(b) If $\hat{S} \subset V$ with $\#(\hat{S}) < \#(\mathcal{S})$ Then $\hat{S}$ does not span $V$.

**NOTES** (1) $\exists$ MSS for $V$
(3) $\exists\ \infty\ \#$ MSS for $V$.

**EX** $\mathcal{S}_1 = \{\vec{e}_1, \vec{e}_2\}$ $\mathcal{S}_2 = \{\vec{e}_1 + \vec{e}_2, \vec{e}_2\}$ are 2 MSS for $\mathbb{R}^2$.

**PROP 5** Any 2 MSS for $V$ contain same # ELTS.

**PF** Let $\mathcal{S}_1, \mathcal{S}_2$ be 2 MSS for $V$
Suppose $\#(\mathcal{S}_1) < \#(\mathcal{S}_2)$.

Since $\mathcal{S}_2$ is a MSS, $\mathcal{S}_1$ cannot span $V$ by (b)

So $\mathcal{S}_1$ cannot be a MSS for $V$ by (a) $\times$

$\square$

**DEF 6** (a) The **DIMENSION** of a FD VS $V$ is # elts in a MSS for $V$

(b) A set $\mathcal{S} = \{\vec{v}_1, \dots, \vec{v}_n\}$ CONTAINS A REDUNDANCY if $\exists j$:

$$\vec{v}_j = \text{L.C. of } \vec{v}_1, \dots, \vec{v}_{j-1}, \vec{v}_{j+1} \dots \vec{v}_n.$$

**REDUNDANCY THM 7**

Let $\mathcal{S}$ be a finite spanning set for $V$.
Then

$\mathcal{S}$ contains a redundancy $\Longleftrightarrow$ $\mathcal{S}$ is NOT a MSS

PF ~~OF~~

$(\Longrightarrow)$

Suppose $\mathcal{S}$ contains a redundancy with $j=n$:

$$\vec{v}_n = \sum_{k=1}^{n-1} \beta_k \vec{v}_k$$

Let $\tilde{\mathcal{S}} = \{\vec{v}_1, \dots, \vec{v}_{n-1}\}$

Then $\tilde{\mathcal{S}}$ spans $V$ too (as in EX 3)

Since $\#(\tilde{\mathcal{S}}) < \#(\mathcal{S})$, $\mathcal{S}$ is NOT a MSS $\square$

$(\Longleftarrow)$ See Later

The def$^n$ of a redundancy is a bit messy, as not "symmetric" in the $\vec{v}_j$'s.
So

DEF 8 (a) A set $\mathcal{S} = \{\vec{v}_1 \dots \vec{v}_n\}$ is LINEARLY DEPT

if $\exists$ scalars $\alpha_1 — \alpha_n$ NOT ALL ZERO :

$$\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \vec{0}.$$

(b) UEV $\mathcal{S}$ is LD $\Leftrightarrow$ $\mathcal{S}$ contains a redundancy.

(c) $\mathcal{S}$ is LINEARLY INDEPT if $\mathcal{S}$ is NOT LD.

(d) UEV $\mathcal{S} = \{\vec{v}_1 \dots \vec{v}_n\}$ is LI if

only sol$^n$ of $\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \vec{0}$

is the trivial sol$^n$ $\alpha_1 = \alpha_2 \dots = \alpha_n = 0$.

(e) A BASIS for a FDVS V is a LI, SPANNING SET for V.

## PROP 9 (USING MATRICES TO TEST FOR LI)

Let $A = [\vec{v}_1 \ldots \vec{v}_n]$ be $m \times n$ So $\vec{v}_j \in \mathbb{R}^m$

TFAE

1. $\{\vec{v}_1 \ldots \vec{v}_n\}$ is a LI set
2. $N(A) = \{0\}$
3. $Rk(A) = n$ ← Use GE on A to check # PIVOTS.

**PF** $\boxed{1 \Leftrightarrow 2}$

$$A\vec{x} \overset{A}{=} \vec{0} \iff \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n \overset{B}{=} \vec{0} \qquad \vec{x} = \begin{pmatrix} \alpha_1 \\ | \\ \alpha_n \end{pmatrix}$$

So

$\{\vec{v}_1 - \vec{v}_n\}$ LI

$\iff$ ONLY SOLN of B is $\alpha_1 = \ldots = \alpha_n = 0$

$\iff$ Only Sol$^n$ of A is $\vec{x} = \vec{0}$

$\iff$ $N(A) = \{\vec{0}\}$

$\boxed{2 \Leftrightarrow 3}$ See L2 Cor 9.

$\square$

## BASIS CHARACTERIZATION THM 10

Let $V$ be a FDVS, $B = \{\vec{b}_1, \dots, \vec{b}_n\} \subseteq V$.

TFAE

1. $B$ is a basis for $V$
2. $B$ is a MSS for $V$
3. $B$ is a MAXIMAL LI set for $V$.

IN PARTICULAR SINCE $V$ has a MSS, $V$ HAS A BASIS!

PF

2 $\Rightarrow$ 1 : See Pf of Red$^y$ Thm 7

1 $\Rightarrow$ 2 : Let $B$ be a basis for $V$.

Suppose $B$ is NOT a MSS.

~~Let~~ So $V = \text{Span}\{\vec{x}_1, \dots, \vec{x}_k\}$ for some $k < n$.

So $\exists A_{ij}$ :

$$\vec{b}_j = \sum_{i=1}^{k} \underset{\underset{k \times n}{\uparrow}}{A_{ij}} \vec{x}_i \qquad j = 1, \dots, n \qquad (*)$$

So $Rk(A) \le \min(k, n) = k < n$

So by L2 Cor 9 $N(A) \ne \{\vec{0}\}$.

So $\exists \vec{z} \in \mathbb{R}^n$ : $A\vec{z} = \vec{0}$.

So $\forall i$

$$0 = (A\vec{z})_i = \sum_{j=1}^{n} A_{ij} z_j \qquad (**)$$

Then

$$\sum_{j=1}^{n} z_j \vec{b}_j = \sum_{j=1}^{n} z_j \left( \sum_{i=1}^{k} A_{ij} \vec{x}_i \right) \quad \text{by } (*)$$

$$= \sum_{i=1}^{k} \left( \sum_{j=1}^{n} z_j A_{ij} \right) \vec{x}_i$$

$$= \sum_{i=1}^{k} \underbrace{(A\vec{z})_i}_{=0} \vec{x}_i = \vec{0} \text{ by } (**)$$

Since $z_j$ not all $0$ $(\vec{z} \neq \vec{0})$

$B = \{\vec{b}_1, \dots \vec{b}_n\}$ is <u>not</u> LI (not a basis)

$\times$

$\square$

$\boxed{(3) \Rightarrow (1)}$

Suppose $B$ is a MAX LI set but $B$ is NOT a basis.

Then $\exists \vec{v} \in V : \vec{v} \notin \operatorname{Span}(B)$ as $B$ does not span $V$.

So by Lemma below

$\{\vec{v}, \vec{b}_1, \ldots, \vec{b}_n\}$ is LI set

So $\mathcal{B}$ is <u>NOT</u> a MAXIMAL LI set $\times$

$\boxed{(1) \Rightarrow (3)}$ SIMILAR IDEAS See [M, p 196]

LEMMA 11

If $\mathcal{B}$ is LI and $\vec{v} \in V$ Then

$$\mathcal{B} \cup \{\vec{v}\} \text{ is LI} \Leftrightarrow \vec{v} \notin \operatorname{Span}(\mathcal{B})$$

PF

$\boxed{\Leftarrow}$ Suppose $\vec{v} \notin \operatorname{Span}(\mathcal{B})$ and

$$\alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n + \gamma \vec{v} = \vec{0} \quad (+)$$

SHOW $\alpha_1 = \cdots = \alpha_n = \gamma = 0$

If $\gamma \neq 0$ Then

$$\vec{v} = -\frac{1}{\gamma}(\alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n) \in \operatorname{Span}(\mathcal{B}) \times$$

So $\gamma = 0$

Then as $\{\vec{b}_1, \ldots, \vec{b}_n\}$ are LI $(+) \Rightarrow \alpha_j = 0 \ \forall j$

$\boxed{\Rightarrow}$ YOU DO IT! $\square$

$\boxed{\Rightarrow}$ Use contrapositive.

Show $\vec{v} \in Span(B) \implies B \cup \{\vec{v}\}$ is LD.

Well

By assumption

$$\vec{v} = \sum_{j=1}^{n} \alpha_j \vec{b}_j$$

So

$$\sum_{j=1}^{n} \alpha_j \vec{b}_j \quad -1\vec{v} = \vec{0}$$

is a nontrivial L.C. of vectors in $B \cup \{\vec{v}\}$ that gives $\vec{0}$.

So $B \cup \{\vec{v}\}$ is LD

D

PROP 12

Suppose $\dim W < \infty$ and $\{\vec{0}\} \neq V \subseteq W$ is a VSS

Then $\dim V < \infty$.

PF Suppose $\dim V = \infty$

CLAIM $\forall k \ \exists \ S_k = \{\vec{v}_1 \dots \vec{v}_k\} \subseteq V$ LI.

PF BY INDUCTION

$\boxed{k=1}$ $V \neq \{\vec{0}\} \Rightarrow \exists \ 0 \neq \vec{v}_1 \in V$.

$\boxed{\text{INDUCTION STEP}}$

Let $S_k = \{\vec{v}_1 \_ \vec{v}_k\} \subseteq V$ be LI

Since $\dim V = \infty$, $S_k$ cannot span $V$

So $\exists \ \vec{v}_{k+1} \notin \mathrm{Span}(S_k)$, $\vec{v}_{k+1} \in V$.

So by LEMMA 11 $S_{k+1}$ is LI. □

By Claim with $k = \dim W + 1$

$\exists$ LI subset of $V$ (and hence of $W$) with $\dim W + 1$ elts.

So any basis for $W$ is NOT a MAXIMALLY LI subset for $W$ $\times$. □

# SUBSPACE DIM THM 13

Let $V, W$ be FD VSs with $V \leq W$
Then

① $\dim V \leq \dim W$

② If $\dim V = \dim W$ Then $V = W$.

PF Let $n = \dim V$

① If $\{\vec{v}_1 \ldots \vec{v}_n\}$ is a basis for $V$

Then $\{\vec{v}_1 \ldots \vec{v}_n\}$ is a LI subset of $V$ and hence of $W$.

So $\dim V = n \leq$ # ELTS in MAX LI set for $W$
$= \dim W$

② If $\dim V = \dim W$ but $V \neq W$

Let $\vec{w} \in W$ with $\vec{w} \notin V$

Let $\{\vec{v}_1 \ldots \vec{v}_n\}$ be basis for $V$

By Lemma 11 $\{\vec{v}_1 \ldots \vec{v}_n, \vec{w}\}$ is LI in $W$

So $n+1 \leq$ MAX # LI elts of $W = \dim W$
So $\dim V = n \leq \dim W - 1$ ✗ $\square$
$< \dim W$

# THM ON BASES FOR 4 FUNDAMENTAL SUBSPACES 14

Let $Rk(A_{m\times n}) = r$ with

$E = PA$ in row echelon form

$\mathcal{H} = \{\vec{h}_1, \ldots, h_{n-r}\}$ set of vectors in GS of $A\vec{x} = \vec{0}$.

Then

① BASIS for $R(A)$ = Set of Basic Cols of $A$

$\dim R(A) = r$

② BASIS for $R(A^T)$ = Set of Nonzero rows of $E$

$\dim R(A^T) = r$

③ BASIS for $N(A) = \mathcal{H}$

$\dim N(A) = n-r$

④ BASIS for $N(A^T)$ = Last $m-r$ rows of $P$

$\dim N(A^T) = m-r$.

PF. • Show Spanning Sets we found in L2 are all LI
• Convince yourself using EXS.

## COR 15 RANK + NULLITY THM

If $A$ is $m \times n$ Then

$$\dim N(A) + \dim R(A) = n$$

### SIMPLEST EX

$$A_{m\times n} = N_r = \begin{bmatrix} I_{r\times r} & O_{r\times(n-r)} \\ O_{(m-r)\times r} & O_{(m-r)\times(n-r)} \end{bmatrix}$$

$\mathbb{R}^n = \text{Span}\{\vec{e}_1 - \vec{e}_n\}$
$\mathbb{R}^m = \text{Span}\{\vec{f}_1 \ldots \vec{f}_m\}$

$N(A) = \text{Span}\{\vec{e}_{r+1} \ldots \vec{e}_n\}$ has dim $n-r$

$R(A) = \text{Span}\{\vec{f}_1 \ldots \vec{f}_r\}$ has dim $r$.

— o —

**PROP 16** Let $V$ be a VSS of $\mathbb{R}^n$

Then $\exists \{\vec{v}_1 \ldots \vec{v}_n\}$ : $V = \text{Span}\{\vec{v}_1 \ldots \vec{v}_n\}$

**PF** By PROP 12 $\dim V < \infty$
So by BASIS CHAR$^n$ THM 10, $V$ has a basis $\square$

**PROP 17** Let $V$ be a VSS of $\mathbb{R}^m$

Then $\exists$ $m \times n$ $A$ : $R(A) = V$

**PF** By PROP 16

$V = \text{Span}\{\vec{v}_1 \ldots \vec{v}_n\}$ for some $n$, $\vec{v}_j \in \mathbb{R}^m$

$= \{\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n \mid \alpha_j \in \mathbb{R}\}$

$= \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$ $\qquad A = [\vec{v}_1 \ldots \vec{v}_n]$

$= R(A)$ $\qquad \blacksquare$

**BASIS EXTN THM 18**

Let $V \subseteq W \subseteq \mathbb{R}^N$
Let $\{\vec{v}_1 \ldots \vec{v}_k\}$ be a basis for $V$
Then $\exists$ vectors $\vec{w}_1 \ldots \vec{w}_\ell$ in $W$ :

$\{\vec{v}_1 \ldots \vec{v}_k, \vec{w}_1 \ldots \vec{w}_\ell\}$ is basis for $W$.

**PF** Let $\{\vec{u}_1 \ldots \vec{u}_{k+\ell}\}$ be any basis for $W$.

Form $A = [\vec{v}_1 \ldots \vec{v}_k \mid \vec{u}_1 \ldots \vec{u}_{k+\ell}]$ $\qquad R(A) = W$ ✓
The basic cols of $A$ give the desired basis. $\blacksquare$