FILTERING + CONVOLUTION

FILTERING is a technique in signal and image processing that smoothes a signal or image. It can

(a) Remove noise from a signal (static)

(b) • Decrease Gibbs overshoot in finite Fourier series approximations of functions with jumps
• Rounds off sharp corners.

(c) Change the shape (sound / look) of the "signal" while removing "noise".

TWO WAYS TO THINK ABOUT FILTERING FOR FOURIER SERIES

(A) FREQUENCY DOMAIN APPROACH

(1) Given $f : [0, 2\pi] \to \mathbb{C}$ that models an audio signal $f = f(t)$ $t =$ TIME write

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

with

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt$$

Since the function $h_k(t) = e^{ikt}$ models a pure tone whose pitch (frequency) is determined by k we call the ~~vector~~ set of c_k's the Frequency Domain representation of the signal.

(2) GUIDING PRINCIPLE

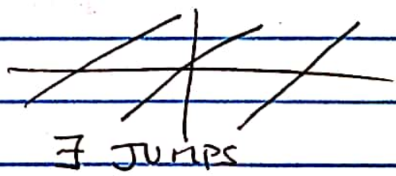
The faster $c_k \to 0$ as $|k| \to \infty$
The smoother the function f is.

REASONS

~~(a) Large~~ $\cancel{c_k}$

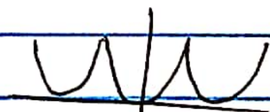
(a) If c_k is large for large $|k|$
Then there is a large ~~osc~~ rapidly oscillating component, $c_k e^{ikx}$ in the signal.

(b) EX $x = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx$



∃ JUMPS

$x^2 = \frac{\pi^2}{3} + 4 \sum_{k=1} \frac{(-1)^k}{k^2} \cos kx$



SMOOTHER AS NO JUMPS

and $\frac{1}{k^2} \to 0$ faster than $\frac{1}{|k|} \to 0$ as $|k| \to \infty$

ASIDE

We can get F.S. for $g(x) = x^2$ from that $f(x) = x$ using

PROP Suppose f is PW CTS with mean 0 on $[-\pi, \pi]$ with

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

Then the F.S. of $g(x) = \int_{-\pi}^{x} f(t)\,dt$ is

$$g(x) = m + \sum_{k=1}^{\infty} -\frac{b_k}{k} \cos kx + \frac{a_k}{k} \sin kx$$

where $m = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\,dx$

"Can integrate term by term".

Using the Guiding Principle ~~let~~ we introduce a FILTER FUNCTION

$$h(x) = \sum_{k=-\infty}^{\infty} d_k e^{ikx}$$

where $d_k \to 0$ fairly fast as $|k| \to \infty$

Then given signal $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$

we use the filter to smooth f by defining

$$f_h(x) = \sum_{k=-\infty}^{\infty} c_k d_k e^{ikx}$$

We will have $c_k d_k \to 0$ faster than $c_k \to 0$

so f_h should be smooth than h.

"High frequency oscillations in f have been damped out by h".

(B) TIME DOMAIN APPROACH

We can also smooth a signal using (local) (weighted) averaging

Suppose we have discretized f on grid $x_0, x_1, \ldots, x_{N-1}$ where $x_j = j\Delta x$

Then we could smooth f using local averaging:

Sf = Smoothed version of f

$$Sf(x_j) = \frac{1}{3}\left[f(x_{j-1}) + f(x_j) + f(x_{j+1})\right]$$

or more generally given weights $\{w_k\}_{k=-K}^{K}$

define

$$(Sf)(x_j) = w_K f(x_{j-K}) + w_{K-1} f(x_{j-K+1}) + \cdots$$

$$+ w_1 f(x_{j-1}) + w_0 f(x_j) + w_{-1} f(x_{j+1})$$

$$+ \cdots + w_{-K+1} f(x_{j+K-1}) + w_{-K} f(x_{j+K})$$

$$= \sum_{k=-K}^{K} w_k f(x_{j-k})$$

Suppose $w_k = h(x_k)$ for some function h
2π-periodic

Then

$$S_h f(x_j) = \sum_{k=-N}^{N} h(x_k) f(x_{j-k})$$

which is a Riemann sum for the

CONVOLUTION of f and h is

$$\boxed{(h * f)(x) := \int_{-\pi}^{\pi} h(y)\, f(x-y)\, dy}$$

SUMMARY

- CONVOLUTION of f by h replaces f by a weighted average of f using weights h
- CONVOLUTION therefore can smooth f

RELATIONSHIP: BETWEEN APPROACHES (A) + (B)

CONVOLUTION THM

Suppose f, h are 2π-periodic and

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}, \quad h(x) = \sum_{k=-\infty}^{\infty} d_k e^{ikx}$$

Then
$$\boxed{(f * h)(x) = 2\pi \sum_{k=-\infty}^{\infty} c_k d_k e^{ikx}}$$

PROOF

NTS $\quad 2\pi \, c_k d_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} (h*f)(x)\, e^{ikx}\, dx$

Well

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (h*f)(x)\, e^{ikx}\, dx$$

$$= \frac{1}{2\pi}\int_{x=-\pi}^{\pi} \left[\int_{y=-\pi}^{\pi} h(y)\, f(x-y)\, dy\right] e^{ikx}\, dx$$

$$= \frac{1}{2\pi}\int_{y=-\pi}^{\pi} h(y) \left[\int_{x=-\pi}^{\pi} f(x-y)\, e^{ik(x-y)}\, dx\right] e^{iky}\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(y) \left(\int_{-\pi}^{\pi} f(u)\, e^{iku}\, du\right) e^{iky}\, dy$$

$$= 2\pi \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} h(y)\, e^{iky}\, dy\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\, e^{iku}\, du\right)$$

$$= 2\pi \quad d_k \quad c_k$$

EXERCISES

① $\quad h*f = f*h$

② If h is differentiable, then $(h*f)' = h'*f$

So $h*f$ is differentiable, even if f is not.

Illustration of Filtering

We filter the 2π-periodic signal defined on $[0, 2\pi]$ by

$$f(t) = \begin{cases} 0, & 0 \leq t < \pi, \\ 1, & \pi \leq t < 2\pi. \end{cases} \tag{1}$$

In some cases we add white noise to f by replacing each value $f(t)$ by $f(t) + X_{0,\mu}(t)$ where the $X_{0,\mu}(t)$ are independent, normally distributed random variables with mean zero and standard deviation μ.

The filter function is chosen to be the Gaussian with standard deviation, σ, defined by

$$h(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\left(\frac{t-\pi}{\sigma}\right)^2\right]. \tag{2}$$

The filtered function is the convolution $(h*f)(t)$, which we compute in the frequency domain using the Convolution Theorem.

All results were obtained by discretizing the time domain with $N = 512$ points and using the DFT.

We first show results with filter width of $\sigma = 0.1$ (moderate filtering).

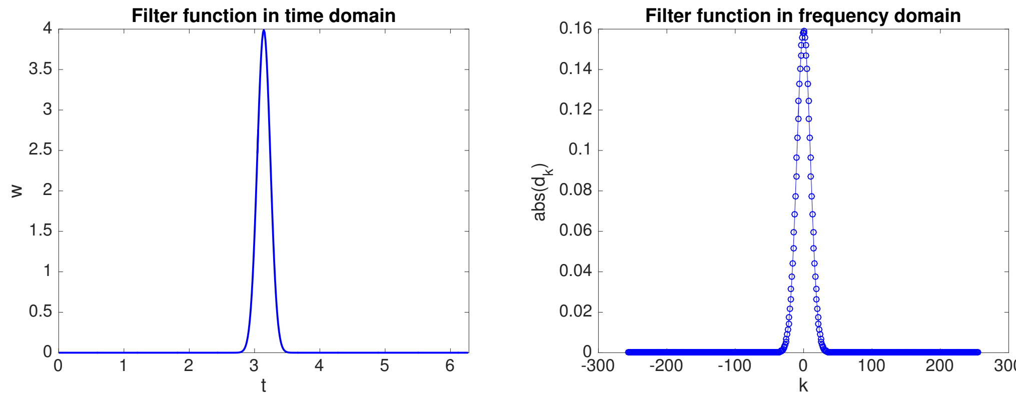


Figure 1: **Amazing Fact:** The Fourier transform of a Gaussian is a Gaussian!

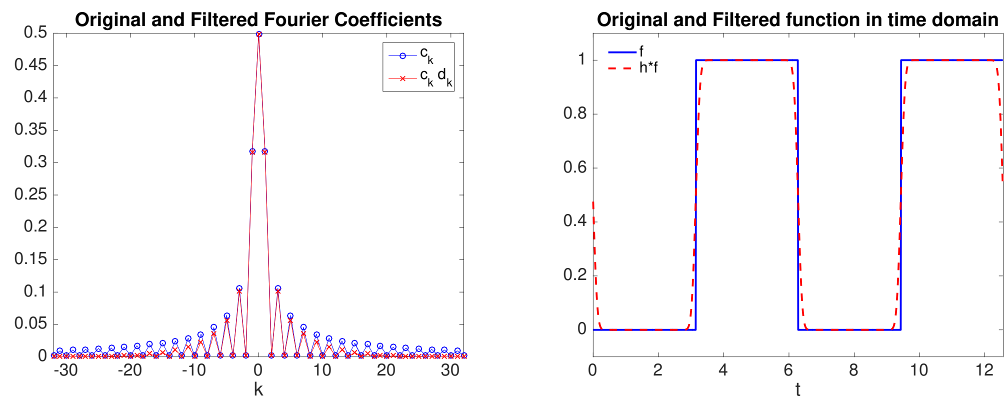


Figure 2: The filter damps high frequencies in f (left) and rounds off sharp corners (smooths) f in the time domain (right).

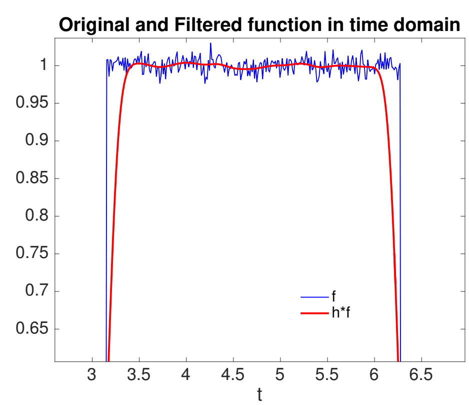


Figure 3: Adding a small amount of noise to f we see that the filter suppresses the high frequency noise, but some low frequency noise remains and signal is smoothed too.

Next we show results with filter width of $\sigma = 0.3$ (strong filtering).

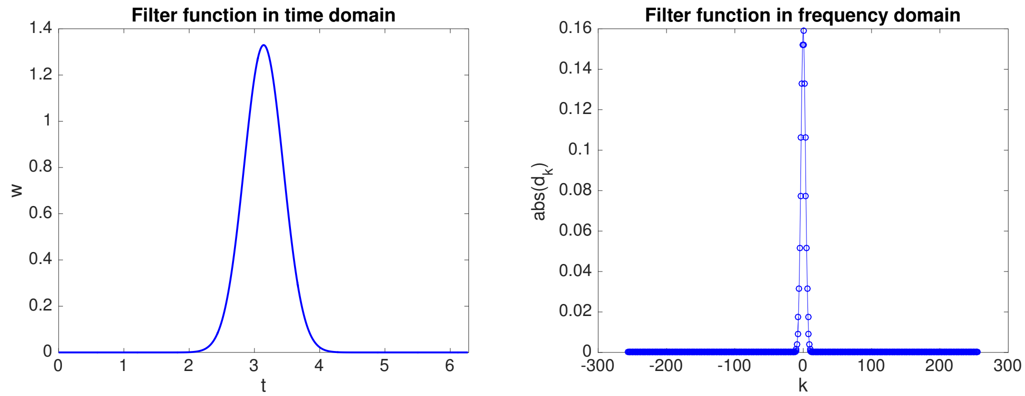


Figure 4: A wider filter in time (more averaging) corresponds to a narrower filter in frequency (more aggresive damping of high frequencies).

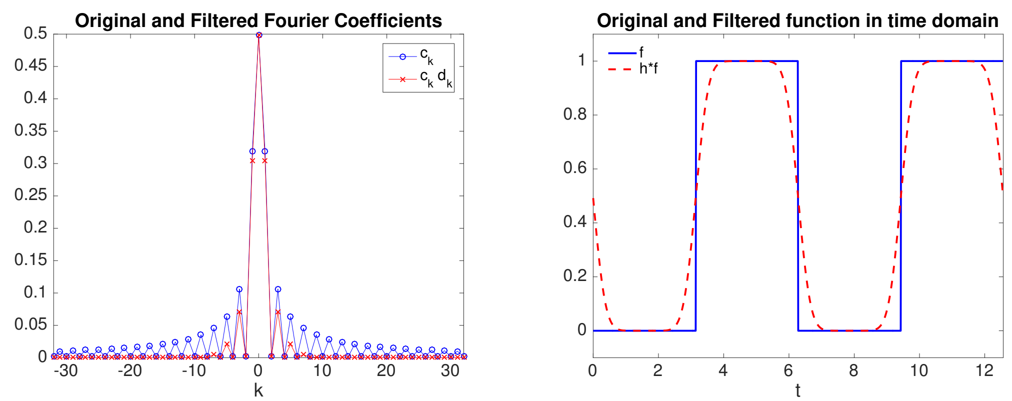


Figure 5: With stronger filtering, high frequencies are damped more but the signal is more distorted and looses it piecewise constant shape.

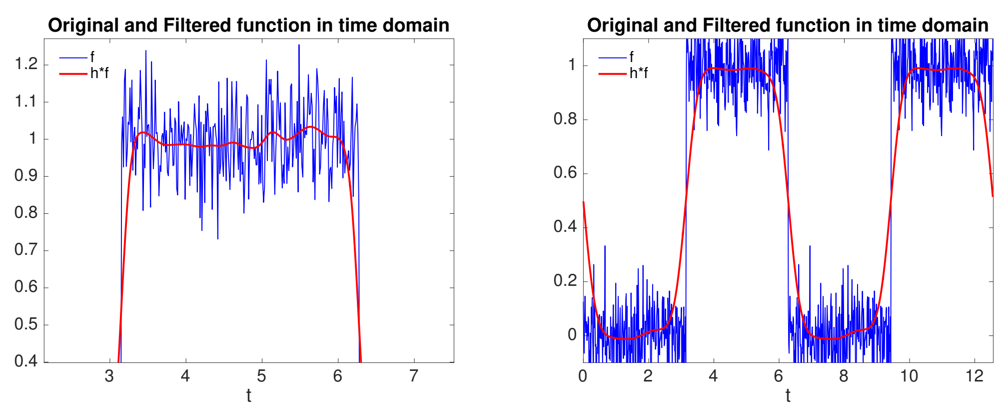


Figure 6: The stronger filter is more effective with large noise (left: $\sigma = 0.1$, right: $\sigma = 0.3$).