THE DISCRETE FOURIER TRANSFORM (DFT)

REFERENCES: Meyer 5.8
Walker 2.1–2.4, 4.6

A DERIVATION OF DFT

Suppose $f : [0, L] \to \mathbb{C}$ is L-periodic

So $f(0) = f(L)$

The $\mathbb{C}$ F.S. of f is

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{i \frac{2\pi k}{L} x} \qquad (1)$$

where

$$c_k = \frac{1}{L} \int_0^L f(x) e^{-i \frac{2\pi k}{L} x} \qquad (2)$$

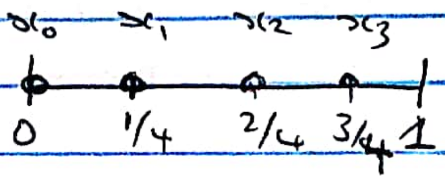
ASIDE: We use $[0, L]$ instead of $[-L, L]$ to make formulae simpler below.

We discretize the interval $[0, L]$ using N points

$$x_j = j \Delta x \qquad j = 0, 1, \ldots, N-1$$

where $\Delta x = \frac{L}{N}$

$\boxed{L = 1, N = 4}$



NOTE

We don't need $x_N = 1$ as $f(0) = f(1)$

Using a Riemann Sum to approximate 2

$$c_k \simeq \frac{1}{L} \sum_{j=0}^{N-1} f(x_j)\, e^{-i \frac{2\pi k}{L} x_j} \Delta x$$

$$= \frac{1}{N} \sum_{j=0}^{N-1} e^{-i \frac{2\pi}{N} jk} u_j \qquad (3A)$$

where

$$\vec{u} = \begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f(x_0) \\ \vdots \\ f(x_{N-1}) \end{pmatrix} \in \mathbb{C}^N \qquad (3B)$$

IN THIS STORY ALL INDICES GO FROM 0 to N−1 INSTEAD OF 1 to N

Define

$$\boxed{\xi = e^{-\frac{2\pi i}{N}}} \qquad (4)$$

DEF 1 Let F be the $N \times N$ matrix with

$$F_{jk} = \xi^{jk}$$

where $0 \le j, k \le N-1$

F IS THE FOURIER MATRIX

2 The DISCRETE FOURIER TRANSFORM (DFT) of is the L.T. $F : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$\vec{V} = F(\vec{u}) = F\vec{u}.$$

So $$v_k = \sum_{j=0}^{N-1} \xi^{jk} u_j \quad \text{for } k = 0, 1, \dots, N-1. \qquad (5)$$

ROOTS OF UNITY

(A) $$\xi = e^{-\frac{2\pi i}{N}} \in \mathbb{C}$$

is an N-th ROOT of UNITY in that

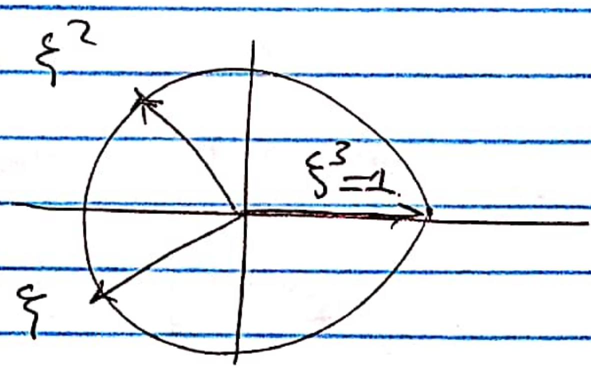
$$\boxed{\xi^N = 1.} \qquad (6)$$

CHECK $$\left(e^{-\frac{2\pi i}{N}}\right)^N = e^{-2\pi i} = 1.$$

(B) ALSO $|\xi| = 1$, ξ lies on UNIT CIRCLE in $\mathbb{C}$.

AND SO $\xi^{-1} = \bar{\xi}$

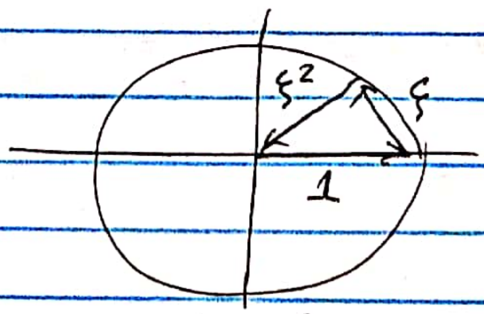
(C) EX $N=3$



$\xi = e^{-i\theta_N}$

$\theta_3 = -2\pi/3$

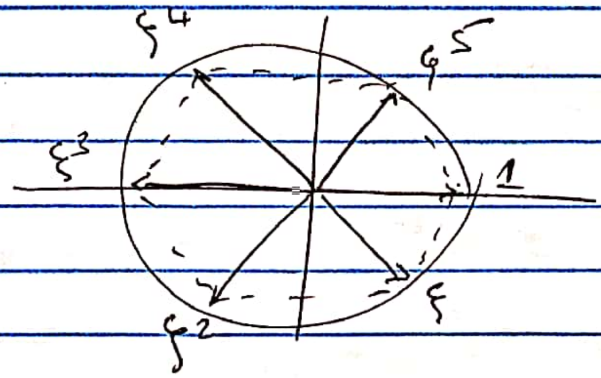
NOTICE



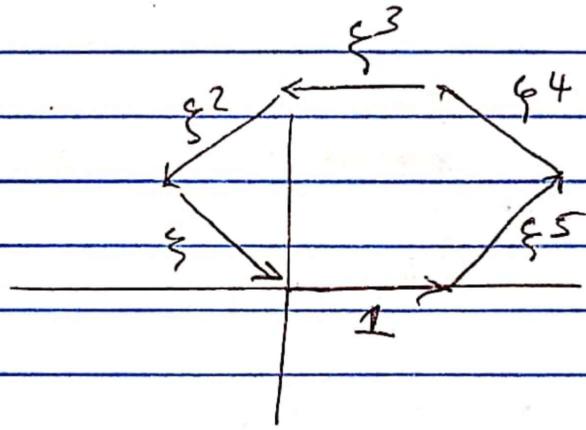
EQUILATERAL Δ.

$$1 + \xi + \xi^2 = 0.$$

N = 6



REGULAR HEXAGON



$$1 + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5 = 0 \quad !!$$

CLAIM FOR $k = 1, 2, \ldots, N-1$ we have

$$1 + \xi^k + (\xi^k)^2 + (\xi^k)^3 + \cdots + (\xi^k)^{N-1} = 0. \quad (7)$$

or $$1 + \xi^k + \xi^{2k} + \xi^{3k} + \cdots + \xi^{(N-1)k} = 0$$

PF

$$\xi^k\left(1 + \xi^k + \xi^{2k} + \cdots + \xi^{(N-1)k}\right)$$

$$= \xi^k + \xi^{2k} + \xi^{3k} + \cdots + \xi^{(N-1)k} + \underbrace{\xi^{Nk}}_{=1}$$

$$= 1 + \xi^k + \cdots + \xi^{(N-1)k}$$

(5)

So
$$\left[1+\xi^{k}+\dots+\xi^{(N-1)k}\right]\left[1-\xi^{k}\right]=0$$

So if $k \neq 0, N$ then $\xi^{k} \neq 1$ and so

$$1+\xi^{k}+\dots+\xi^{(N-1)k}=0 \qquad \square$$

MORE ON THE FOURIER MATRIX

Recall $F_{jk}=\xi^{jk}$ $\qquad 0 \le j,k \le N-1$

So

$$F=\begin{array}{c}\left[\begin{array}{ccccc}1 & 1 & 1 & \cdots & 1\\ 1 & \xi & \xi^{2} & \cdots & \xi^{N-1}\\ 1 & \xi^{2} & \xi^{4} & \cdots & \xi^{N-2}\\ \vdots & & & & \\ 1 & \xi^{N-1} & \xi^{N-2} & \cdots & \xi\end{array}\right]\begin{array}{l}\leftarrow \text{ROW } 0\\ \\ \\ \\ \leftarrow \text{ROW } N-1\end{array}\\ \begin{array}{ccccc}\uparrow & & & & \uparrow\\ \text{COL } 0 & & & & \text{COL } N-1\end{array}\end{array} \qquad (8)$$

Row j has entries $1, \xi^{j}, \xi^{2j}, \dots, \xi^{(N-1)j}$

NOTICE (a) $(2, N-1)$-ENTRY $=(\xi^{2})^{N-1}=\xi^{2N-2}=\xi^{N}\xi^{N-2}=\xi^{N-2}$

(b) $(N-1, N-1)$-ENTRY $=(\xi^{N-1})^{N-1}=\xi^{N(N-1)}\xi^{1-N}=\xi$

(c) $\boxed{N=4}$ $(2,2)$-ENTRY $=(\xi^{2})^{2}=\xi^{4}=1$.

CLAIM

1. $F^T = F$

2. $F^* F = N I = F F^*$

3. So $U = \frac{1}{\sqrt{N}} F$ is UNITARY:

$$U U^* = I = U^* U$$

DEF The INVERSE DFT is $F^{-1} : \mathbb{C}^N \to \mathbb{C}^N$ given by

- $\vec{u} = F^{-1}(\vec{v})$
- $\vec{u} = \frac{1}{N} F^* \vec{v}$
- $u_j = \frac{1}{N} \sum_{k=0}^{N-1} \zeta^{-jk} v_k$ (9)

PROOF OF 3

(a) $(F^*)_{jk} = \overline{F_{kj}} = \overline{\zeta^{jk}} = \zeta^{-jk}$

(b)

$$(F^* F)_{jk} = \sum_{l=0}^{N-1} F^*_{jl} F_{lk}$$

$$= \sum_{l=0}^{N-1} \zeta^{-jl} \zeta^{lk} = \sum_{l=0}^{N-1} \left(\zeta^{k-j}\right)^l$$

$\boxed{\text{CASE } k \neq j}$

$$0 \leq k \leq N-1$$
$$-(N-1) \leq -j \leq 0$$

$$-(N-1) \leq k-j \leq N-1$$

So $\quad 0 \neq k-j \neq \pm N \implies \xi^{k-j} \neq 1$

So $\quad (F^*F)_{jk} = \sum_{l=0}^{N-1} (\xi^{k-j})^l = 0 \quad$ by (8)

$\boxed{\text{CASE } k = j}$ $\quad (F^*F)_{kk} = \sum_{l=0}^{N-1} 1^l = N \quad$ as $\xi^0 = 1$

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RELATION BETWEEN DFT AND FOURIER SERIES COEFFICIENTS

Recall $\quad c_k \sim \frac{1}{N} V_k \quad$ where

$$c_k = \frac{1}{L}\int_0^L f(x)\, e^{-i\frac{2\pi k}{L}x}\, dx \qquad \forall k \in \mathbb{Z}$$

and $\quad V_k \overset{(*)}{=} \sum_{j=0}^{N-1} \xi^{jk} u_j \quad$ was originally defined for $k = 0, 1, \ldots, N-1$.

But (*) makes sense $\forall k \in \mathbb{Z}$, and

$\forall k \qquad V_{k+N} = V_k \qquad$ as $\quad \xi^{jN} = 1^j = 1.$

PROBLEM BUT THEN $c_{k+N} \simeq \frac{1}{N} v_{k+N} = \frac{1}{N} v_k \simeq c_k$ 8

which is obviously false.

RULE OF THUMB

$$c_k \simeq \frac{1}{N} v_k \quad \text{for} \quad |k| < \frac{N}{8}$$

PUT ANOTHER WAY

Suppose given $f : [0, L] \to \mathbb{C}$ periodic and given k, and want to approximate c_k using DFT. If so do:

1. Pick $N > 8|k|$
2. Sample f at the N points $x_j = \frac{jL}{N}$ to get u_j
3. Calculate $\vec{v} = F\vec{u}$ (ie Do Riemann Sums)
4. Then $c_k \sim \frac{1}{N} v_k$ will be a good approximation.

THE FAST FOURIER TRANSFORM (FFT)

1. COMPUTATIONAL COST OF $\vec{v} = F\vec{u}$ IS $O(N^2)$ as have N inner products each $O(N)$.
2. USING SPECIAL STRUCTURE OF F CAN SPEED THIS UP TO $O(N \log N)$ which is much faster when N is large.

Computation of Fourier coefficients using DFT

The sawtooth function is the 2π-periodic extension of the function $f(x) = x$ on the interval $[-\pi, \pi]$. We know from lectures on Fourier series that it's Fourier coefficients are given by

$$c_k = \begin{cases} \frac{(-1)^k}{k} i & k > 0 \\ 0 & k = 0 \\ \frac{(-1)^{k+1}}{k} i & k < 0 \end{cases} \tag{1}$$

We can also pick number of points, N, for the DFT and estimate c_k using the method on page 8 of the DFT lecture. In the figure legends that method is indicated with the blue circles. The exact answers given by the equation for c_k above is given by the red crosses.

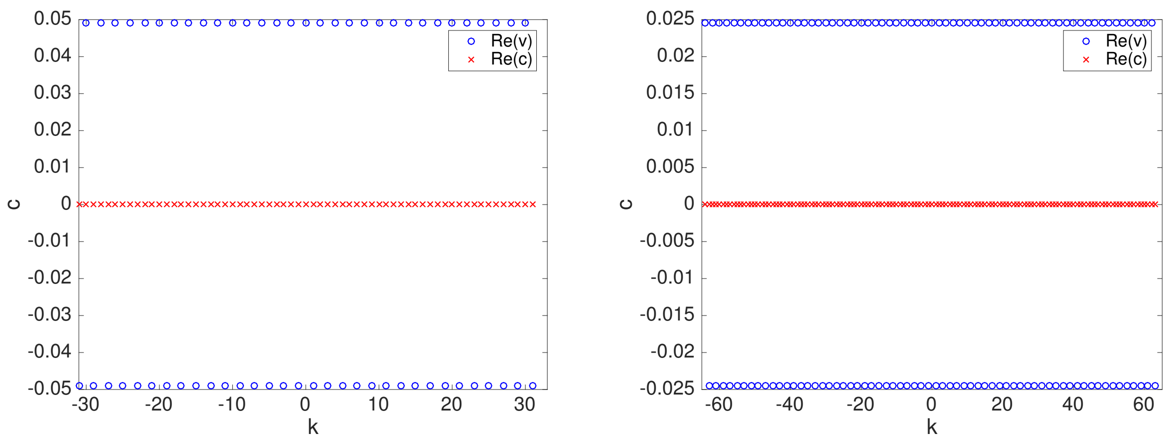

Figure 1: Left: $N = 64$, Right: $N = 128$.

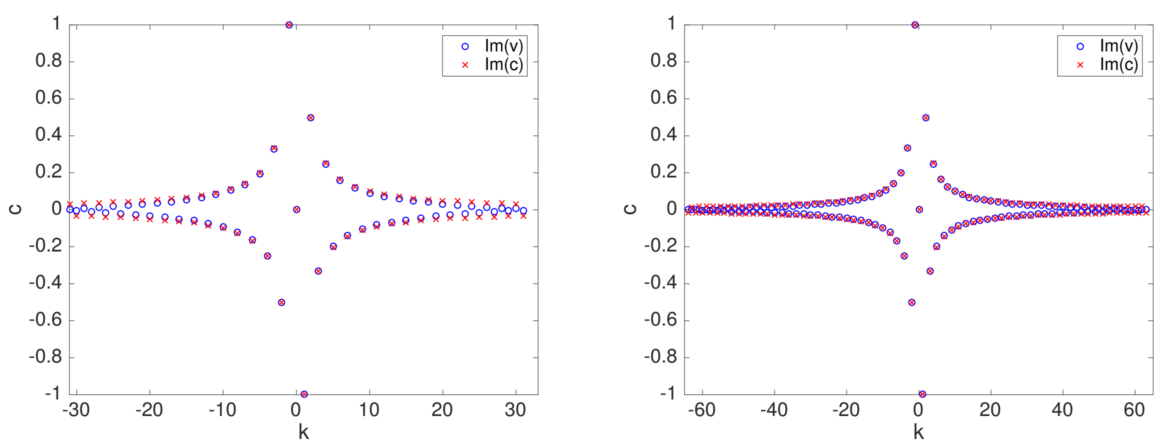

Figure 2: Left: $N = 64$, Right: $N = 128$.

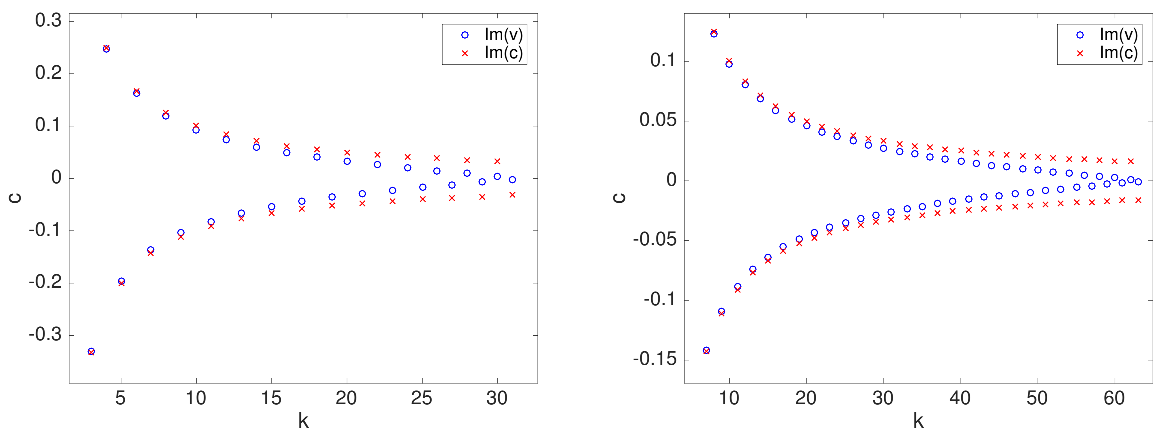

Figure 3: Left: $N = 64$, Right: $N = 128$.