FOURIER SERIES IV : UNIFORM CONVERGENCE + DIFFERENT'.

DEF 1 Let $u_k : [a,b] \to \mathbb{R}$, $k = 1, 2, 3, \ldots$

and $S_n(x) = \sum_{k=1}^{n} u_k(x)$

ⓐ We say $\sum_{k=1}^{\infty} u_k$ converges $\left\{\begin{matrix} \text{POINTWISE} \\ \text{UNIFORMLY} \end{matrix}\right\}$ IF

$$S_n \to S := \sum_{k=1}^{\infty} u_k \quad \left\{\begin{matrix} \text{POINTWISE} \\ \text{UNIFORMLY} \end{matrix}\right\}$$

ⓑ We say $\sum u_k$ converges ABSOLUTELY if

$\sum |u_k|$ converges

THM 2
WEIERSTRASS M-TEST

Suppose each u_k is bounded in that

$\exists m_k : \quad |u_k(x)| \leq m_k \quad \forall x \in [a, b]$,

If $\sum_{k=1}^{\infty} m_k$ converges then

$\sum_{k=1}^{\infty} u_k(x) =: f(x)$ CONVERGES UNIFORMLY + ABSOLUTELY

Hence if each u_k is CTS, so is f.

EX3 Fourier Series for x^2:

$$x^2 = \frac{\pi^2}{3} + 4\pi \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx) \quad (*)$$

Choose $m_k = \frac{1}{k^2}$ so that

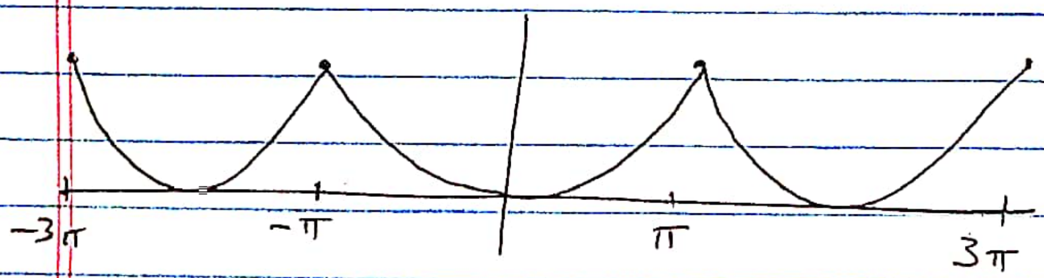
$u_k(x) = \frac{(-1)^k}{k^2} \cos(kx)$ has

$|u_k(x)| \le m_k$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ we have

F.S $(*)$ converges uniformly and absolutely.

NOTE The periodic extension of $f(x) = x^2$ is CTS



We can manipulate uniformly convergent series just like finite sums.

THM 4 If $f(x) = \sum_{k=1}^{\infty} u_k(x)$ converges UNIFORMLY

Then ① For any bounded f^n g

$$g(x) f(x) = \sum_{k=1}^{\infty} g(x) u_k(x)$$

② $$\int_a^b f(y)\,dy = \int_a^b \sum_{k=1}^{\infty} u_k(y)\,dy$$

$$= \sum_{k=1}^{\infty} \int_a^b u_k(y)\,dy.$$

– Can integrate series term by term.

EX 5

$$x^2 = \frac{\pi^2}{3} + 4\pi \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx)$$

converges uniformly. So

$$x^3 = 3\int_0^x t^2\,dt = 3\int_0^x \left[\frac{\pi^2}{3} + 4\pi \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kt)\right] dt$$

$$= 3\left[\frac{\pi^2}{3}x + 4\pi \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \int_0^x \cos(kt)\,dt \right]$$

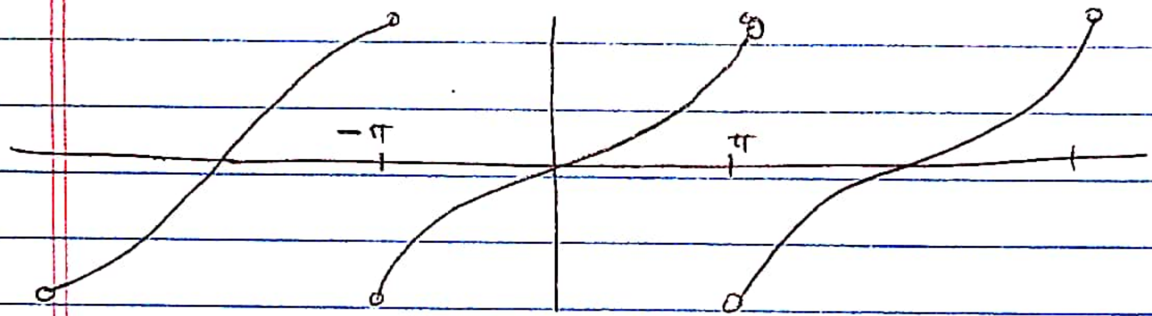
$$= \pi^2 x + 4\pi \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \frac{1}{k} \sin kx$$

NOW

$$x = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx) \quad \text{from before}$$

So

$$x^3 = \sum_{k=1}^{\infty} \left[\frac{2\pi^2(-1)^{k+1}}{k} + \frac{4\pi(-1)^k}{k^3} \right] \sin(kx)$$



CONVERGENCE IS ONLY POINTWISE HERE AS f_{PER} has JUMPS.

— o —

THM 6 Suppose $f(x) = \sum_{k=1}^{\infty} u_k(x)$ POINTWISE

and $\sum_{k=1}^{\infty} u_k'(x) =: g(x)$ UNIFORMLY

Then
$f(x) = \sum_{k=1}^{\infty} u_k(x)$ converges UNIFORMLY and

$$f' = g$$

ie

$$\frac{d}{dx}\left(\sum_{k=1}^{\infty} u_k(x) \right) = \sum_{k=1}^{\infty} \frac{du_k}{dx}(x)$$

Ex 7 Why do we need uniform convergence of series of derivatives?

Well $$x = 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin(kx) \qquad \text{PW}$$

but taking d/dx gives

$$1 = 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k} k\cos(kx)$$

BUT $$1 \neq 2\sum_{k=1}^{\infty} (-1)^k \cos(kx).$$

THIS SERIES DIVERGES for most x.

THM 8 **DIFFERENT^n OF F.S.**

Suppose $f:[-\pi,\pi] \to \mathbb{R}$ has a CTS, 2π-PERIODIC piecewise C^2 extension to $f_{PER} : \mathbb{R} \to \mathbb{R}$

If $$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

Then $$f'(x) \approx \sum_{k=1}^{\infty} -k a_k \sin kx + k b_k \cos kx \quad (*)$$

NOTE f PW $C^2 \Rightarrow f'$ PW C^1 so $(*)$ CONVERGES PW.

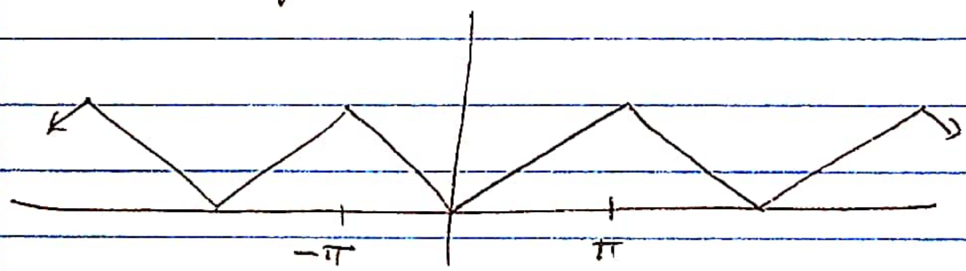
<u>NOTE</u> With $f(x) = x$, f_{PER} is <u>NOT</u> CTS.

So Thm 8 does not apply!

<u>PROOF</u> Combine FS II Thm 12 with Thm 6 above □

<u>EX 9</u> Let $f(x) = |x|$ on $[-\pi, \pi]$

Periodic extⁿ of f is CTS, 2π-periodic, PW C^2



Can show

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right]$$

$$\frac{d}{dx}|x| = \text{sign}(x) \sim \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

is true ✓.

~~cont~~

THM 10

Suppose $f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$

IF $\sum_{k=-\infty}^{\infty} |c_k| < \infty$

Then the F.S. CONVS UNIFORMLY to a CTS f^n $\tilde{f}$.

and

$$c_k = \langle \tilde{f}, e^{ikx} \rangle = \langle f, e^{ikx} \rangle$$

ie $f, \tilde{f}$ have same Fourier Coefficients

PF

$|c_k e^{ikx}| = c_k$ so FS converges UNIF by Weierstrass M-Test.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f}(x) e^{-ilx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_k c_k e^{ikx} \right] e^{-ilx} dx$$

THM 4(1)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_k c_k e^{i(k-l)x} dx$$

THM 4(2)

$$= \frac{1}{2\pi} \sum_k \int_{-\pi}^{\pi} c_k e^{i(k-l)x} dx$$

$$= \frac{1}{2\pi} \sum_k c_k \delta_{kl} = c_l.$$

$\square$

NOTE Any function f like sawtooth function for which Fourier coefficients have $|c_k| \propto \frac{1}{k}$ will not satisfy Thm 10.

THM 11

Suppose $f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$

Then

1. $\sum_{k=-\infty}^{\infty} |c_k|^2 \leq \|f\|^2$ BESSEL'S INEQUALITY

2. Hence if $\|f\| < \infty$ then

$$c_k \to 0 \quad \text{as} \quad k \to \pm\infty.$$

ALSO $a_k \to 0$, $b_k \to 0$ as $k \to \infty$,

as
$$a_k = c_k + c_{-k}$$
$$b_k = i(c_k - c_{-k})$$

META PRINCIPLE

THE SMOOTHER f IS THE FASTER $c_k \to 0$

— INTUITION: Fewer $\uparrow$ frequency oscillations

Pf

Let $\varphi_k(x) = e^{ikx}$

$$S_n = \sum_{|k| \le n} c_k \varphi_k .$$

Now

(A) $\|S_n\|^2 = \langle S_n, S_n \rangle$

$$= \left\langle \sum_{|k| \le n} c_k \varphi_k , \sum_{|l| \le n} c_l \varphi_l \right\rangle$$

$$= \sum_{|k|, |l| \le n} c_k \overline{c_l} \langle \varphi_k, \varphi_l \rangle$$

$$= \sum_{|k|, |l| \le n} c_k \overline{c_l} \delta_{kl} = \sum_{|k| \le n} |c_k|^2$$

$$\boxed{\|S_n\|^2 \le \sum_{|k| \le n} |c_k|^2}$$

(B) $\langle f, S_n \rangle = \left\langle f, \sum_{|k| \le n} c_k \varphi_k \right\rangle$

$$= \sum_{|k| \le n} \overline{c_k} \langle f, \varphi_k \rangle$$

$$= \sum_{|k| \le n} |c_k|^2 = \|S_n\|^2$$

(C) $0 \le \|f - S_n\|^2$

$$= \|f\|^2 + \|S_n\|^2 - 2\langle f, S_n\rangle$$

$$\overset{(A),(B)}{=} \|f\|^2 - \sum_{|k| \le n} |c_k|^2$$

So

$$\|S_n\|^2 = \sum_{|k| \le n} |c_k|^2 \le \|f\|^2 \quad \forall n$$

(D) So $\sum_{k=-\infty}^{\infty} |c_k|^2 = \lim_{n\to\infty} \|S_n\|^2$ converges

as $\|S_n\|^2$ is $\uparrow$ seq bounded above by $\|f\|^2$

Hence $c_k \to 0$ must hold as $k \to \pm\infty$.

$\boxed{\text{PARSEVAL'S IDENTITY}}$

Suppose $\|f\| = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx} < \infty$.

Then

$$\boxed{\|f\|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2}$$