FOURIER SERIES III: SCALING AND COMPLEX FOURIER SERIES

(A) CHANGE OF SCALE

Let $f : [-L, L] \to \mathbb{R}$

Define $s : [-\pi, \pi] \to [-L, L]$ by

$$s(y) = \frac{L}{\pi} y$$

Let

$F : [-\pi, \pi] \to \mathbb{R}$ be $F = f \circ s$

$$F(y) = f(s(y)) = f\left(\frac{L}{\pi} y\right) = f(x) \qquad \boxed{x = \frac{L}{\pi} y}$$

We have

$$F(y) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(ky) + b_k \sin(ky)$$

with

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \cos(ky)\, dy$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \sin(ky)\, dy$$

$\boxed{\text{So}}$

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)$$

with

$$a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi x}{L}\right) dx \qquad \text{VIA CofV}$$

$$dy = \frac{\pi}{L} dx$$

$$b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{k\pi x}{L}\right) dx$$

B) COMPLEX FOURIER SERIES

EULER'S FORMULA

$$e^{ikx} = \cos(kx) + i\sin(kx)$$

$$e^{-ikx} = \cos(kx) - i\sin(kx)$$

So

$$\cos(kx) = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$$

$$\sin(kx) = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$$

So instead of ~~the~~ expressing $f = f(x)$ in terms of cos, sin we could use $e^{\pm ikx}$'s.

But we need to work with the COMPLEX IPS

$$C^0([-\pi,\pi],\mathbb{C}) = \{ f : [-\pi,\pi] \xrightarrow{CB} \mathbb{C} \}$$

with

$$\langle f, g \rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx$$

NOTE DIFFERENT NORMALIZ^{n}

L^2 NORM $\quad \|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx}$.

PROP 1 $\{e^{ikx} \mid k \in \mathbb{Z}\}$ is an ~~ONS~~ ~~fa~~ ON set in $C^0([-\pi,\pi], \mathbb{C})$

PF

① $\|e^{ikx}\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx} e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dx = 1$

– This explains new normalizⁿ above

② IF $l \neq k$

$$\langle e^{ikx}, e^{ilx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\,\overline{e^{ilx}}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-l)x}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{i(k-l)x}}{i(k-l)}\right]_{-\pi}^{\pi} = 0$$

$\square$

DEF 2 Let $f:[-\pi,\pi] \to \mathbb{C}$.

The COMPLEX F.S. of f is

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

where

$$c_k = \langle f, e^{ikx} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx.$$

$\boxed{\text{RELATION TO } a_k, b_k}$

$$a_k \cos kx + b_k \sin kx$$

$$= \frac{1}{2} a_k \left(e^{ikx} + e^{-ikx}\right) + \frac{1}{2i} b_k \left(e^{ikx} - e^{-ikx}\right)$$

$$= \frac{1}{2}(a_k - ib_k)\, e^{ikx} + \frac{1}{2}(a_k + ib_k)\, e^{-ikx}$$

UPSHOT
$k>0$ $\quad \mathrm{Span}\{\cos kx, \sin kx\} = \mathrm{Span}\{e^{-ikx}, e^{ikx}\}$

and

$$c_k = \frac{1}{2}(a_k - ib_k) \quad \text{for } k>0$$

$$c_{-k} = \frac{1}{2}(a_k + ib_k) \quad \text{for } k>0.$$

$$c_0 = \frac{a_0}{2}.$$

EX $f(x)=e^{ax}$, $a\neq 0$ $\quad a\in\mathbb{R}$

$$c_k=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-ikx}\,dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-ik)x}\,dx$$

$$=\frac{1}{2\pi}\left[\frac{e^{(a-ik)x}}{a-ik}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2}\,\frac{e^{(a-ik)\pi}-e^{-(a-ik)\pi}}{(a-ik)\pi}$$

$$=(-1)^k\,\frac{e^{a\pi}-e^{-a\pi}}{2\pi(a-ik)} \qquad \text{as } e^{ik\pi}=(-1)^k$$

$$=(-1)^k\,\frac{(a+ik)\sinh(a\pi)}{a^2+k^2} \qquad \text{as } \sinh(x)=\frac{e^{x}-e^{-x}}{2}$$

So

$$e^{ax}\sim\frac{\sinh(a\pi)}{\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^k(a+ik)}{a^2+k^2}e^{ikx}$$

- NOTE THAT THIS SERIES CONVERGES SLOWLY
- f_{PER} IS NOT CTS
- BIG JUMPS FOR LARGE a.