L2 SUMMARY OF THEORY ON SOLVING $A\vec{x}=\vec{b}$

A ROW OPERATIONS + ROW ECHELON FORM [M, 2.1], [M, 3.9]

FACT 1 ① Using elementary row operations any $m \times n$ matrix A can be converted ~~to~~ a ROW ECHELON FORM, E:

$$E = \begin{bmatrix} \circledast & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \circledast & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & 0 & \circledast & * & * & \cdots & \cdots \\ \vdots & \vdots & \vdots & & & & \circledast & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

STAIRCASE STRUCTURE.

THE PIVOT entries are first non-zero entries in each row (Circled)

② The PIVOT positions are uniquely determined by A.

DEF 2 ① The BASIC COLS of A are those cols of A that contain pivot positions.

② The rank of A is

$$\begin{aligned} \text{Rank}(A) &= \#\ \text{PIVOTS} \\ &= \#\ \text{NON-ZERO ROWS of } A \\ &= \#\ \text{BASIC COLS in } A \end{aligned}$$

NOTE $\text{Rank}(A) \leq \min\{n, m\}$.

REDUCED ROW ECHELON FORM

$E_{m\times n}$ is in Reduced REF if

- E is in REF
- 1st nonzero entry in each row is a 1
- All entries above each pivot is 0.

$$\begin{pmatrix} \boxed{1} & * & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & \boxed{1} & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

FACTS

① For any $A_{m\times n}$ $\exists!$ RREF, denoted E_A.

② EACH NONBASIC COL IN E_A IS A L.C. OF BASIC COLS IN E_A TO LEFT OF THE GIVEN COL

EG
$$\begin{pmatrix} * \\ * \\ * \\ * \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

③ EXACTLY SAME RELATIONSHIPS $\exists$ AMONG COLS OF A AS AMONG COLS OF E_A (i.e. same coefficients)

$$A_{*8} = \alpha_1 A_{*1} + \alpha_2 A_{*3} + \alpha_3 A_{*6} + \alpha_4 A_{*7}$$

DEF 3 There are 3 types of elementary row operations. Each corresponds to left multiplication by a $m \times m$ matrix that is invertible

We illustrate these using examples

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$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

I $R2 \Longleftrightarrow R1$

$$\begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$U = E_{12} \quad A$$

Since $I = E_{12} E_{12} I = E_{12} E_{12}$ we know E_{12} is INV.

II $R1 \to \alpha R1$, $\alpha \neq 0$

$$\begin{pmatrix} \alpha & 2\alpha & 3\alpha \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$U = F_{\alpha} \quad A$$

Since $F_{\frac{1}{\alpha}} F_{\alpha} = I$, F_{α} is INV.

III $R2 \to R2 + \alpha R1$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4+\alpha & 5+2\alpha & 6+3\alpha \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$U = G_\alpha \quad A$$

Since $G_{-\alpha} G_\alpha = I$ $\quad G_\alpha$ is INV.

NOTE E_{ij}, F_α, G_α are called ELEMENTARY MATRICES

PROP 4

① Suppose E is a row echelon form of A.
Then $\exists$ INV P so that

$$E = PA.$$

P will be a product of matrices of form E_{ij}, F_α, G_α.

② Suppose there is a linear relation among the cols of A
Then the same relation must hold among the ~~rows~~ cols of E

$$\boxed{(PA)_{*k} = PA_{*k}}$$

EG If $A_{*3} = 2A_{*1} + 4A_{*2}$

Then $PA_{*3} = 2PA_{*1} + 4PA_{*2}$

$(PA)_{*3} = 2(PA)_{*1} + 4(PA)_{*2}$

So $E_{*3} = 2E_{*1} + 4E_{*2}$ ✓

B THE NULLSPACE OF A [M, 2.4] [M, 4.2]

DEF 5 Given $m \times n$ A defining $F: \mathbb{R}^n \to \mathbb{R}^m$ by $F(\vec{x}) = A\vec{x}$ the NULLSPACE of A is the solution set of $A\vec{x} = \vec{0}$:

$$N(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$$

PROP 6

(a) $\vec{0} \in N(A) \neq \emptyset$

(b) $N(A)$ is a VSS of $\mathbb{R}^n$

(c) Let $E = PA$ be a row echelon form of A. Then $N(A) = N(E)$.

PF

(b) Let $\alpha \in \mathbb{R}$, $\vec{x}_1, \vec{x}_2 \in N(A)$.
Then $\alpha\vec{x}_1 + \vec{x}_2 \in N(A)$ since
$A(\alpha\vec{x}_1 + \vec{x}_2) = \alpha A\vec{x}_1 + A\vec{x}_2 = \alpha\vec{0} + \vec{0} = \vec{0}$.

(c) Show $A\vec{x} = \vec{0} \Longleftrightarrow E\vec{x} = \vec{0}$

$\boxed{\Rightarrow}$ Suppose $A\vec{x} = \vec{0}$. Then $E\vec{x} = (PA)\vec{x} = P(A\vec{x}) = \vec{0}$

$\boxed{\Leftarrow}$ Use $A = P^{-1}E$

$\square$

Def 7

a) A variable x_j is BASIC if A_{*j} is a basic col.

b) A variable is FREE if it is not basic.

So

$r = \#$ Basic Vars $= Rk(A)$

$n - r = \#$ Free Vars.

HOW TO CALCULATE $N(A) = N(E)$

EX Suppose E is row echelon form of A. and

$E\vec{x} = \vec{0}$ is

$$\begin{bmatrix} \boxed{2} & 3 & 4 & 6 & 7 \\ 0 & \boxed{1} & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & \boxed{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve $E\vec{x} = \vec{0}$ use back substitution to express basic variables in terms of free ones.

BASIC VARS : x_1, x_2, x_5 $\qquad r = 3$

FREE VARS : x_3, x_4 $\qquad n - r = 2$

Get

$x_5 = 0$

$x_2 + 4x_5 = 0 \Rightarrow x_2 = 0$

$2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 = 0 \Rightarrow x_1 = -2x_3 - 3x_4$

OR

$$\vec{x} = \begin{bmatrix} -2x_3 - 3x_4 \\ 0 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= x_3 \vec{h}_1 + x_4 \vec{h}_2$$

So

$$N(A) = N(E) = \operatorname{SPAN}\{\vec{h}_1, \vec{h}_2\} \subseteq \mathbb{R}^5$$

IN GENERAL we GET:

PROP 8 The solution set of $A\vec{x} = \vec{0}$ is set of $\vec{x} \in \mathbb{R}^n$ of form

$$\vec{x} = x_{f_1} \vec{h}_1 + \cdots + x_{f_{n-r}} \vec{h}_{n-r}$$

where $r = Rk(A) = \#$ Basic Variables

$n - r = \#$ free Vars

$x_{f_j} = j$-TH free Var $\quad \vec{h}_{f_j} \in \mathbb{R}^n$.

ie

$$N(A) = \operatorname{SPAN}\{\vec{h}_1, \ldots, \vec{h}_{n-r}\}$$

COR 9 If A is $m \times n$

$$N(A) = \{0\} \iff Rk(A) = n$$

C THE SOLUTION SET OF $A\vec{x}=\vec{b}$ AND RANGE(A) [M 2.5, 4.2]

When we use Gaussian Elimination to solve $A\vec{x}=\vec{b}$
we row reduce $[A|\vec{b}]$ to $[E|\vec{c}]$
where E is a row echelon form for A

CLAIM 10

$$\vec{x} \text{ solves } A\vec{x}=\vec{b} \iff \vec{x} \text{ solves } E\vec{x}=\vec{c}$$

PROOF $\exists$ INV P : $P[A|\vec{b}]=[E|\vec{c}]$
So $PA=E$, $P\vec{b}=\vec{c}$

Then $A\vec{x}=\vec{b} \iff PA\vec{x}=P\vec{b}$ (as P inv)
$\iff E\vec{x}=\vec{c}$. ∎

POTENTIAL PROBLEM : $A\vec{x}=\vec{b}$ might not have any solutions.

EX Suppose get

$$[E|\vec{c}] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 3 \end{array}\right] \iff \left\{\begin{array}{l} x_1 = 2 \\ 0 = 3 \end{array}\right\} \quad \text{NO SOLNS}$$

DEF 11 $A\vec{x}=\vec{b}$ is CONSISTENT if $\exists$ at least one solution $\vec{x}$.

PROP 12 $A\vec{x} = \vec{b}$ is consistent iff none of rows of $[E | \vec{c}\,]$ are ~~of~~ of form $(0 — 0 | \alpha)$ for $\alpha \neq 0$.

PROP 13 Suppose $A\vec{x} = \vec{b}$ is consistent

Let $\vec{p}$ be a particular sol^n of $A\vec{x} = \vec{b}$

Then any solution is of form

$$\vec{x} = \vec{p} + \vec{n} \qquad \text{where } \vec{n} \in N(A)$$

PF $A\vec{x} = \vec{b}$

$\iff A\vec{x} - A\vec{p} = \vec{b} - \vec{b} = \vec{0}$

$\iff A(\vec{x} - \vec{p}) = \vec{0}$

$\iff \vec{x} - \vec{p} \in N(A)$

$\iff \vec{x} = \vec{p} + \vec{n}$ for some $\vec{n} \in N(A)$ $\square$

QN How do we identify those $\vec{b}$ for which $A\vec{x} = \vec{b}$ is CONSISTENT?

DEF 14 Let $F : \mathbb{R}^n \to \mathbb{R}^m$ where $F(\vec{x}) = A\vec{x}$ (A is $m \times n$)

$$R(A) = \text{RANGE}(A) := \{ \vec{y} \in \mathbb{R}^m / \vec{y} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n \}$$

So $R(A)$ is set of $\vec{b} \in \mathbb{R}^m$ for which $A\vec{x} = \vec{b}$ is consistent.

PROP 14

$R(A)$ is a VSS of $\mathbb{R}^m$.

PF (a) $0 \in R(A)$ as $A\vec{0} = \vec{0}$. So $R(A) \neq \emptyset$

(b) Let $\alpha \in \mathbb{R}$, $\vec{y}_1, \vec{y}_2 \in R(A)$

So $\vec{y}_j = A\vec{x}_j$ for some $\vec{x}_j \in \mathbb{R}^n$.

Then
$$\begin{aligned} \alpha \vec{y}_1 + \vec{y}_2 &= \alpha A\vec{x}_1 + A\vec{x}_2 \\ &= A(\alpha \vec{x}_1 + \vec{x}_2) \\ &= A\vec{x} \quad \text{for some } \vec{x} \in \mathbb{R}^n \end{aligned}$$

So $\alpha \vec{y}_1 + \vec{y}_2 \in R(A)$ ✓ $\square$

PROP 15

$R(A)$ = Span of Basic Cols in A.

PF Recall if $A = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ $\vec{x} = \begin{pmatrix} \alpha_1 \\ | \\ \alpha_n \end{pmatrix}$
Then

$$\vec{b} = A\vec{x} = \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n$$

So
$$\vec{b} \in R(A) \iff \vec{b} = \sum_{j=1}^{n} \alpha_j \vec{v}_j \quad \text{for some } \alpha_j$$
$$\iff \vec{b} \in \mathrm{SPAN}\{\vec{v}_1, \ldots, \vec{v}_n\}$$
$$\iff \vec{b} \in \text{SPAN OF BASIC COLS OF } A$$

since each non-basic col of A is a linear combination of the basic cols of A

which, by PROP 4 (2), follows from

<u>CLAIM 16</u> Each non-basic col of E is a L.C of the basic cols of E

<u>PF OF CLAIM BY EX</u>

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 5 & 9 \\ 3 & 7 & 13 \end{pmatrix} \rightarrow E = \begin{pmatrix} \boxed{1} & 3 & 5 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

BASIC BASIC   BASIC BASIC

Clearly Col 3 of E is a LC of cols 1, 2.
In fact

$$E_{*3} = E_{*2} + 2E_{*1}$$

So $\quad A_{*3} = 2A_{*1} + A_{*2}$ must hold ✓

So

$$R(A) = \text{SPAN}\{A_{*1}, A_{*2}\} = \text{SPAN}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\5\\7\end{pmatrix}\right\}$$

D

MATRIX INVERSION THM [M, 3.7]

Let A be $n \times n$

THM 17 The Following are Equivalent:

1. A is invertible
2. $Rk(A) = n$
3. $N(A) = \{0\}$
4. Gauss – Jordan Elimination reduces $[A|I]$ to $[I|A^{-1}]$
5. $\det(A) \neq 0$.
6. A is a product of elementary matrices

$\boxed{\text{PF OF } 4 \Rightarrow 6}$

$$[I|A^{-1}] = P[A|I] \quad \text{where } P = \text{Product of Elementary Matrices}$$
$$= [PA|P]$$

So $PA = I$ ~~and~~ $A^{-1} = P$ as req'd $\square$.

E

RANK NORMAL FORM [M, 3.9]

PROP 18 B is obtained from A by Elementary Col Operat^ns

$\Leftrightarrow$ $B = AQ$ where Q is INV.

PF $B \overset{COL}{\sim} A \Leftrightarrow B^T \overset{ROW}{\sim} A^T$ $\qquad (P^T)^{-1} = (P^{-1})^T$

$\Leftrightarrow B^T = PA^T$ $\quad$ P INV

$\Leftrightarrow B = AP^T \Leftrightarrow B = AQ$ $\quad$ Q INV

DEF 19 $B \sim A$ means B can be obtained from A by a combination of EROs and ECOs.

NOTE $\sim$ IS AN EQUIVALENCE RELATION!

PROP 20 $B \sim A \iff \exists$ INV P, Q: $B = PAQ$

THM 21 [RANK NORMAL FORM]

Let A be $m \times n$ with $Rk(A) = r$.
Then

$$A \sim N_r = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

PF BY EX

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 9 \\ 5 & 10 & 12 \end{pmatrix} \overset{\text{ROW}}{\sim} \begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{-3} \\ 0 & 0 & 0 \end{pmatrix}$$

A ROW ECHELON FORM

$\boxed{r = 2}$

$$\overset{\text{ROW}}{\sim} \begin{pmatrix} \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$$

Reduced RE FORM

$$\overset{\text{COL}}{\sim} \left(\begin{array}{cc|c} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

MOVE ALL PIVOTS TO FAR LEFT TO GET I_r

(13)

$$\overset{\text{COL}}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$ Make last $n-r$ COLS ZERO.

$= N_2.$

$\square$

COR 22 $B \sim A \iff Rk(B) = Rk(A)$

PF

$\boxed{\Leftarrow}$ $B \sim N_r$, $A \sim N_r$ So $B \sim A$

$\boxed{\Rightarrow}$ $Rk(A) = r$, $Rk(B) = s$

$\Rightarrow N_r \sim A \sim B \sim N_s$

$\Rightarrow r = s$ must hold.

COR 23 $Rk(A^T) = Rk(A)$

PF Let $r = Rk(A)$

So

$PAQ = N_r$ with P, Q INV $\Rightarrow (PAQ)^T = (N_r)^T$

$\Rightarrow Q^T A^T P^T = N_r$

$\Rightarrow A^T \sim N_r$ as Q^T, P^T INV

$\Rightarrow Rk(A^T) = r$

$\square$.

$\boxed{F}$ FOUR FUNDAMENTAL SUBSPACES OF $m \times n$ A [M, 4.2]

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\vec{x} \longmapsto A\vec{x}$$

$$G : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$
$$\vec{y} \longmapsto A^T\vec{y}$$

$N(A)$, $R(A^T)$ are VSSs of $\mathbb{R}^n$
$R(A)$, $N(A^T)$ are VSSs of $\mathbb{R}^m$

$\boxed{R(A^T)}$

$$R(A^T) = \{ A^T\vec{y} \ / \ \vec{y} \in \mathbb{R}^m \}$$
$$= \{ \beta_1 (A^T)_{*1} + \dots \beta_m (A^T)_{*m} \ / \ \beta_j \in \mathbb{R} \}$$
$$= \{ \beta_1 A_{1*} + \dots + \beta_m A_{m*} \ / \ \beta_j \in \mathbb{R} \}$$
$$= \text{Span Rows of } A$$

PROP ~~23~~ 24

Let E be a row echelon form of A
Then
$$R(A^T) = \text{Span of non-zero rows of } E.$$

EX (AS ABOVE)
$$R(A^T) = \text{Span}\{(1,3,5),\ (0,1,1)\}$$

PF $R(A^T) = \{ A^T \vec{y} \ / \ \vec{y} \in \mathbb{R}^m \}$

$= \{ (P^{-1}E)^T \vec{y} \ / \ \vec{y} \in \mathbb{R}^m \}$

$= \{ E^T (P^T)^{-1} \vec{y} \ / \ \vec{y} \in \mathbb{R}^m \}$

$= \{ E^T \vec{z} \ / \ \vec{z} \in \mathbb{R}^m \}$ $\quad \vec{z} = (P^T)^{-1} \vec{y}$

Every $\vec{z}$ in $\mathbb{R}^m$ is of this form as ~~PE~~ set $\vec{y} = P^T \vec{z}$.

$\overset{\text{AS ABOVE}}{=}$ Span of Rows of E

$=$ Span of Non-zero Rows of E

$\square$

$\boxed{N(A^T)}$

$N(A^T) = \{ \vec{y} \in \mathbb{R}^m \ / \ A^T \vec{y} = \vec{0} \}$

$= \{ \vec{y} \in \mathbb{R}^m \ / \ \vec{y}^T A = \vec{0} \}$ "LEFT-HAND NULL SPACE OF A"

THM 25 Let A be $m \times n$

Let $r = Rk(A)$ and $E = \underset{m \times m}{P}A$ be an Row Echelon Form of A.

Then LAST $m - r$ ROWS of P SPAN $N(A^T)$

ie If $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \begin{matrix} r \times m \\ (m-r) \times m \end{matrix}$ Then $N(A^T) = R(P_2^T)$.

EX $(A \mid I) \overset{\text{ROW}}{\sim} (E \mid P)$

$$\underbrace{\left(\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{array}\right.}_{A} \left|\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)}_{I} \overset{\text{ROW}}{\sim} \underbrace{\left(\begin{array}{cccc} \boxed{1} & 2 & 1 & 1 \\ 0 & 0 & 0 & \boxed{1} \\ \hline 0 & 0 & 0 & 0 \end{array}\right.}_{E} \left|\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ -3 & 0 & 1 \\ \hline -2 & 1 & 0 \end{array}\right)}_{P}$$

$m = 3, \ r = 2.$

$P_2 = (-2, 1, 0)$

So $N(A^T) = R(P_2^T) = \text{SPAN}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right\}$

CHECK

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

PARTIAL PF We will show $R(P_2^T) \subseteq N(A^T)$.

$$E_{m \times n} = \begin{bmatrix} C_{r \times n} \\ O_{(m-r) \times n} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} A = \begin{pmatrix} P_1 A \\ P_2 A \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix}$$

Let $\vec{v} \in R(P_2^T)$

So $\vec{v} = P_2^T \vec{w}$

Then $A^T \vec{v} = A^T P_2^T \vec{w} = (P_2 A)^T \vec{w} = 0$

So $\vec{v} \in N(A^T)$ ∎