THM 8

Let $f_{PER}: \mathbb{R} \to \mathbb{R}$ be the 2π-periodic extension of a C^1 function ~~of~~ $f: [-\pi, \pi) \to \mathbb{R}$

Then f_{PER} is PIECEWISE C^1

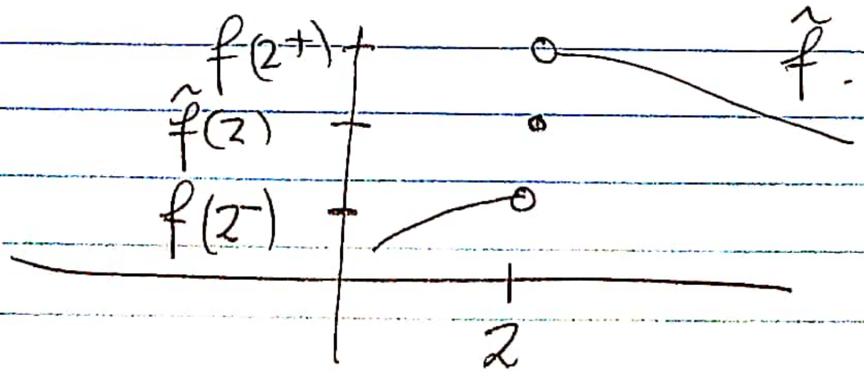
Generalizing Thm 2:

THM 9 [POINTWISE CONVERGENCE THM FOR F.S.]

Let $f: \mathbb{R} \to \mathbb{R}$ be 2π-periodic, piecewise C^1

Then $\forall x \in \mathbb{R}$ the Fourier Series of f converges to

$$\tilde{f}(x) = \begin{cases} f(x) & \text{IF } f \text{ CTS AT } x \\ \frac{1}{2}\left[f(x^+) + f(x^-)\right] & \text{IF } f \text{ has JUMP DISCTY AT } x \end{cases}$$



EX 9 $\boxed{\text{SAWTOOTH FUNCTION}}$

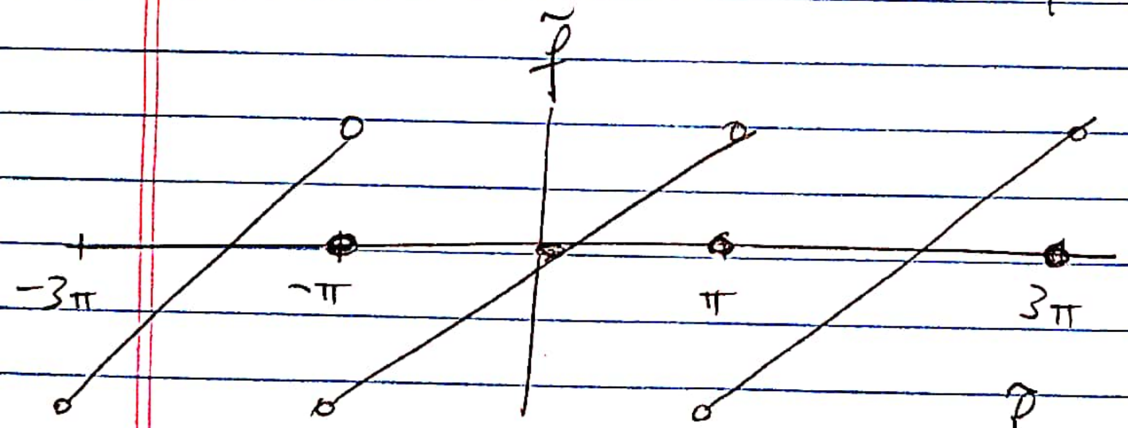
Let $f(x) = x$ on $[-\pi, \pi]$.

We know

$$x \sim 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx) =: \hat{f}(x)$$

So Thm 9 says

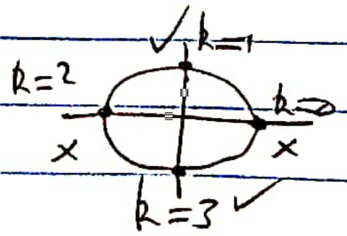
$$2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx) = \begin{cases} x - 2m\pi & \text{IF } (2m-1)\pi < x < (2m+1)\pi \\ 0 & \text{IF } x = (2n-1)\pi \text{ for some } n. \end{cases}$$



$\tilde{f}$ is 2π–PERIODIC.

NOTES (0) AT $x=0$, all partial sums are zero, SO $S_N(0) = \tilde{f}(0)$.

(1) At $x = \pi/2$ we have



$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin\left(\frac{k\pi}{2}\right)$$

$k = 2m+1$

$\sin\left(\frac{k\pi}{2}\right) = (-1)^m$

So
$$\frac{\pi}{4} = \sum_{m=0}^{\infty} \frac{(-1)^{2m+2}}{2m+1} (-1)^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}$$
$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

AN ALTERNATING SERIES

By Alt Series Thm

$$E_N = \left| S_N(\pi/2) - \tilde{f}(\pi/2) \right| \le \frac{1}{2M+1} \quad \text{SLOW CONVERGE.}$$

By way of comparison the series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \quad \text{Converges Very Fast}$$

By Taylor Series Remainder Thm in This case

$$E_N = \left| e - \sum_{n=0}^{N} \frac{1}{n!} \right| \le \frac{3^{N+1}}{(N+1)!}$$

N	E_N $\pi/4$ SERIES	E_N e SERIES
10^1	2.26×10^{-2}	10^{-2}
10^2	2.47×10^{-3}	10^{-111}
10^3	2.5×10^{-4}	?

2 (a) Let $x = \pi - \delta$ for small $\delta > 0$.
We know

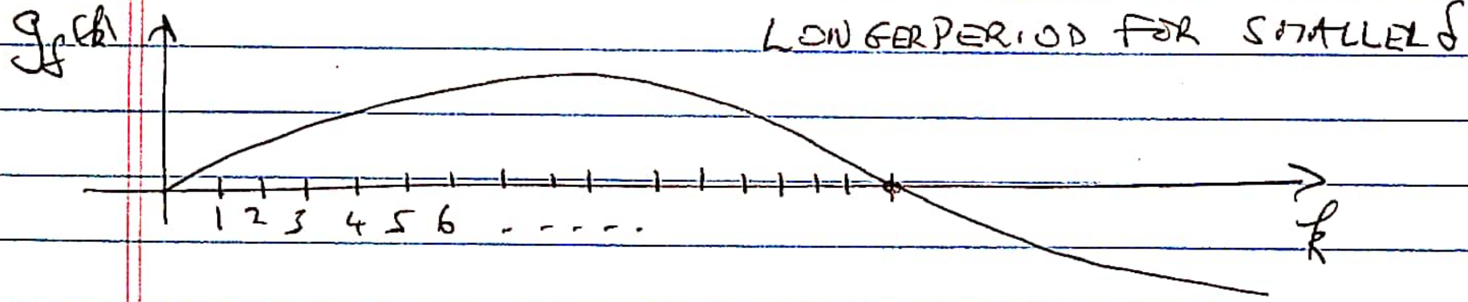
$$\pi - \delta = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(k\pi - k\delta)$$

$$= 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[\cancel{\sin k\pi} \cos k\delta - \underbrace{\cos k\pi}_{(-1)^k} \sin k\delta \right]$$

$$= 2 \sum_{k=1}^{\infty} \frac{\sin k\delta}{k}$$

(b)

Let $g_\delta(k) = \sin(\delta k)$, period $\frac{2\pi}{\delta}$.



So for small δ, have long strings of terms $\frac{\sin k\delta}{k}$ with the same sign

If truncate series near end of string of +ve (or −ve) terms series will behave like the divergent harmonic series $\sum_k \frac{1}{k}$ (or $-\sum_k \frac{1}{k}$).

(c) So $S_N(\pi - \delta)$ will be somewhat greater (or less) than $\pi - \delta$

and $S_N(\pi - \delta) \longrightarrow \pi - \delta$ <u>VERY</u> SLOWLY for small δ.

(d) Since have longer strings of +ve (or −ve) terms for smaller δ, the convergence is slower the closer x is to π

<u>MORE RIGOROUSLY</u>: The convergence of $S_N(x) \to \tilde{f}(x)$ is POINTWISE but <u>NOT</u> UNIFORM in x.

<u>DEF 10</u>

(1) $f_n \to f$ POINTWISE ON $[a,b]$ if $\forall x \in [a,b]$

$\forall \varepsilon > 0 \ \exists N = \underline{\underline{N(\varepsilon, x)}}$: $|f_n(x) - f(x)| < \varepsilon$

SMALL ERROR

(2) $f_n \to f$ UNIFORMLY ON $[a,b]$ if

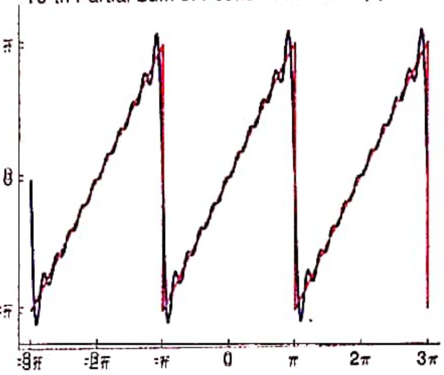
$\forall \varepsilon > 0 \ \exists N = \underline{\underline{N(\varepsilon)}}$: $\forall x \in [a,b]$ $|f_n(x) - f(x)| < \varepsilon$

Scanned with CamScanner

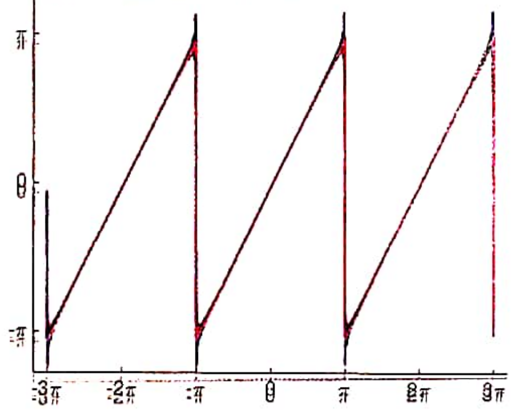
3 17

Fourier Series of Sawtooth Function and Gibbs Phenomenon

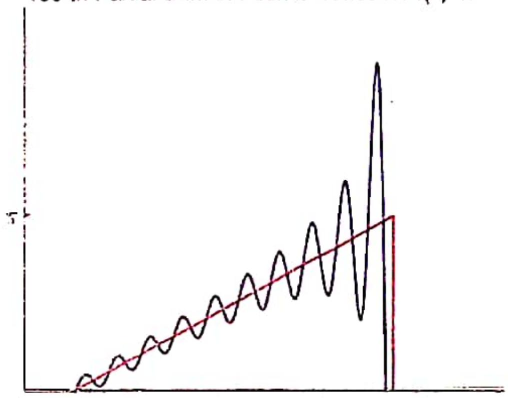
10-th Partial Sum of Fourier Series for f(x)=x



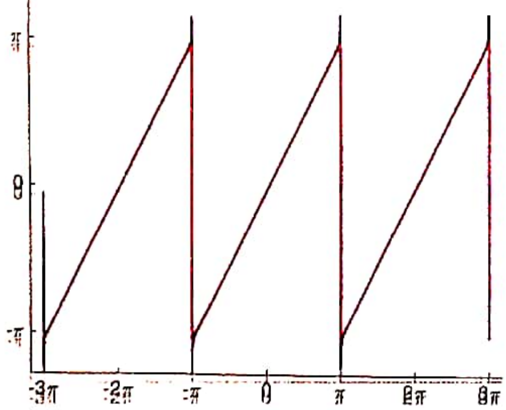
100-th Partial Sum of Fourier Series for f(x)=x



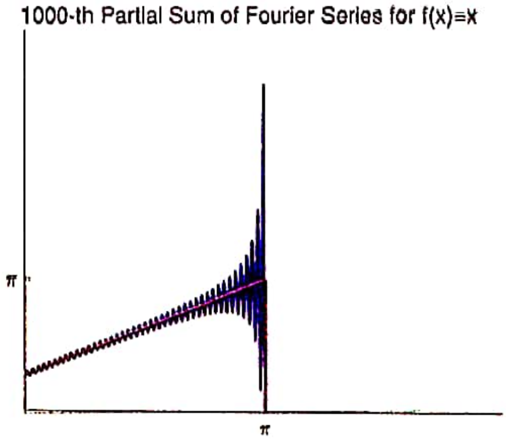
100-th Partial Sum of Fourier Series for f(x)=x



1000-th Partial Sum of Fourier Series for f(x)=x



1000-th Partial Sum of Fourier Series for f(x)=x



THM 11

If $f_n \to f$ UNIFORMLY on $[a,b]$ with each f_n CTS on $[a,b]$
Then f is CTS on $[a,b]$.

Hence For the sawtooth function $S_N \to \tilde{f}$ cannot be UNIFORM on $[-\pi,\pi]$ as $\tilde{f}$ is not CTS at $x=\pi$

MORE SPECIFICALLY FOR FOURIER SERIES :

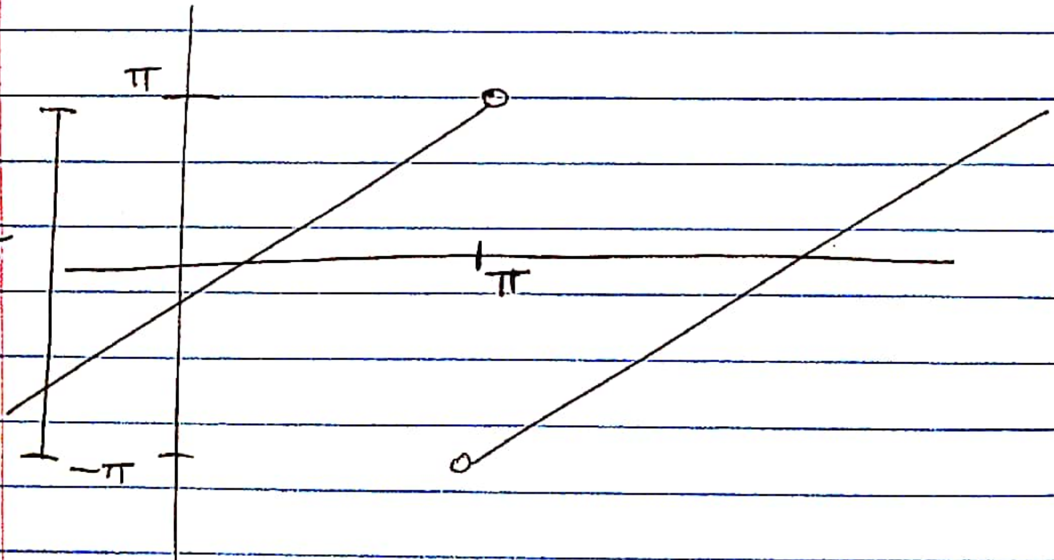
GIBB'S PHENOMENON

Suppose f has a JUMP DISCONTINUITY at x_0 with JUMP MAGNITUDE $D = |f(x_0^+) - f(x_0^-)|$

EX

$x_0 = \pi$

$D = 2\pi$



$f(x) = x$
OW $[-\pi,\pi]$.

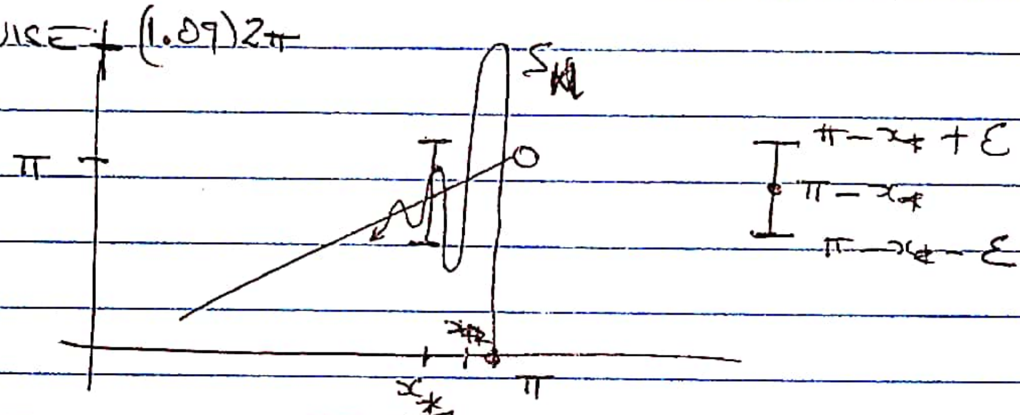
Scanned with CamScanner

- No matter the value of N, near x_0

  $S_N(x)$ overshoots $\tilde{f}(x)$ by $\sim 9\%$ of D.

- But as $N \to \infty$, the width of overshoot region decreases to allow $S_N(x) \to \hat{f}(x)$

POINTWISE



~~Given $\varepsilon > 0$~~

~~and~~ Given $x_* < \pi$ $_\wedge$ [AND $\varepsilon > 0$] $\exists\ N = N(\varepsilon, x_*) : \forall n > N$

$$|S_n(x_*) - (\pi - x_*)| < \varepsilon$$

But $\exists\ x_{**} \in [x_*, \pi]$ so that

$$|S_n(x_{**}) - (\pi - x_{**})| \simeq 1.09D > \varepsilon$$

As x_* moves closer to π, $N(\varepsilon, x_*) \uparrow$

but still have an x_{**} as above with the 9% overshoot.

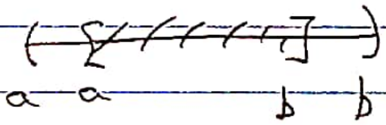
Some Good News

THM 12 (NO PROOF)

Let f be 2π-periodic, piecewise C^1 on $\mathbb{R}$.

If f is CTS on (a,b)

Then $\forall \delta > 0$ $S_N \to f$ UNIFORMLY ON $[a+\delta, b-\delta]$



EX Have uniform convergence of sawtooth FS on $[-\pi+\delta, \pi-\delta]$.

THINK Convergence is slowest at the 2 endpoints.