FOURIER SERIES II : CONVERGENCE

DEF 1 Let $f : \mathbb{R} \to \mathbb{R}$

(a) f is C^1 if f is differentiable and f' is CTS.

(b) f is 2π-PERIODIC if

$$f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}$$

(c) The n-TH PARTIAL SUM of Fourier Series of f is

$$S_n(x) = \frac{a_0}{2} + \sum_{k=0}^{n} a_k \cos(kx) + b_k \sin(kx)$$

We say the F.S. of f CONVERGES at x if

$$\lim_{n\to\infty} S_n(x) = \tilde{f}(x) \quad \text{EXISTS.}$$

ie $\forall \varepsilon > 0 \ \exists N : \forall n \geq N \quad |S_n(x) - \tilde{f}(x)| < \varepsilon$

Here $N = N(\varepsilon, x)$. (POINTWISE CONVERGENCE)

THM 2. Let $f : \mathbb{R} \to \mathbb{R}$ be C^1 and 2π-PERIODIC

Then $\forall x \in \mathbb{R}$

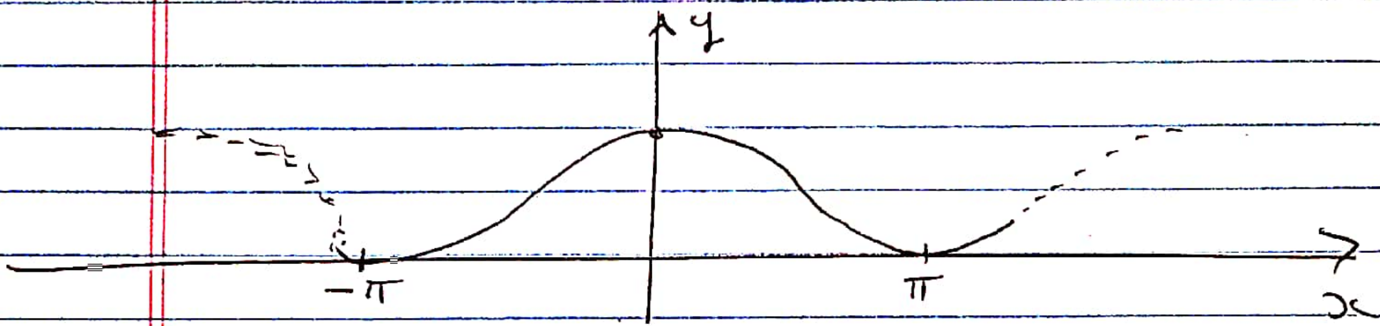
$$f(x) = \frac{a_0}{2} + \sum_{k=0}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

"F.S. OF f CONVERGES TO f POINTWISE"

EX 3

Let $f(x) = (x^2 - \pi^2)^2$ for $-\pi \le x \le \pi$

Extend f to a 2π-periodic function on $\mathbb{R}$



CLAIM: f is C^1 on $\mathbb{R}$.

PF
- $f(-\pi) = f(\pi) = 0$. So f is CTS.
- $f'(x) = 2(x^2 - \pi^2) \cdot 2x$

So $f'(-\pi) = f(\pi) = 0$ So f' is CTS

(Left + Right derivatives of periodic extension are both zero at $x = k\pi$)

ASIDE f is simplest 2π-periodic, C^1, function I could come up with ~~based on~~ that is a polynomial.

Fourier Series of $f(x) = (x^2 - \pi^2)^2 = x^4 - 2\pi^2 x^2 + \pi^4$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} (x^4 - 2\pi^2 x^2 + \pi^4)\, dx$$

$$= \frac{\pi^4}{5} - \frac{2}{3}\pi^4 + \pi^4 = \frac{8}{15}\pi^4.$$

Since f is even, $b_k = 0 \ \forall k.$

for $k \geq 1$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 \cos(kx)\, dx - 2\pi^2 \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(kx)\, dx + 0.$$

Using Integration by Parts formulae

$$\int u^n \cos u\, du = u^n \sin u - n\int u^{n-1} \sin u\, du$$

$$\int u^n \sin u\, du = -u^n \cos u + n\int u^{n-1} \cos u\, du$$

we get

$$\int u^4 \cos u\, du = u^4 \sin u + 4u^3 \cos u - 12\int u^2 \cos u\, du$$

$$\int u^2 \cos u\, du = u^2 \sin u + 2u \cos u - 2\sin u.$$

So

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 \cos(kx)\,dx \qquad u = kx, \quad du = k\,dx$$

$$= \frac{2}{\pi}\int_0^{k\pi} \left(\frac{u}{k}\right)^4 \cos(u)\,\frac{du}{k}$$

$$= \frac{2}{\pi k^5}\int_0^{k\pi} u^4 \cos u\,du$$

$$= \frac{2}{\pi k^5}\Big[u^4 \sin u + 4u^3 \cos u\Big]_0^{k\pi} - \frac{24}{\pi k^5}\int_0^{k\pi} u^2 \cos u\,du$$

$$= \frac{8\pi^2}{k^2}(-1)^k - \frac{24}{\pi k^5}\Big[u^2 \sin u + 2u\cos u - 2\sin u\Big]_0^{k\pi}$$

$$= \left[\frac{8\pi^2}{k^2} - \frac{48}{k^4}\right](-1)^k$$

USING $\cos(k\pi) = (-1)^k$.

CANCEL : MAGIC

Similarly

$$-2\pi^2 \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(kx)\,dx = -2\pi^2 \frac{4}{k^2}(-1)^k = -\frac{8\pi^2}{k^2}(-1)^k$$

So for $k \geq 1$

$$a_k = -48\,\frac{(-1)^k}{k^4} \qquad \text{and}$$

on $[-\pi, \pi]$

$$\boxed{(x^2-\pi^2)^2 \overset{\text{TMN}}{=} \frac{8}{15}\pi^4 - 48\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4}\cos(kx)} \quad (*)$$

C^1

CONVERGES RAPIDLY.

NOTES ① $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4} \cos(kx)$ converges absolutely

as

$$\left| \frac{(-1)^k}{k^4} \cos(kx) \right| \leq \frac{1}{k^4}$$

and $\sum_{k=1}^{\infty} \frac{1}{k^4}$ converges

② If f_{PER} is periodic extension of

$f(x) = (x^2 - \pi^2)^2$ from $[-\pi, \pi]$ to $\mathbb{R}$

Then

$$f_{PER}(x) = \frac{8}{15}\pi^4 - 48 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^4} \cos(kx)$$

for all $x \in \mathbb{R}$

MATLAB EXERCISE A

Plot f_{PER} on $[-2\pi, 2\pi]$ as well as

$$S_N(x) = \frac{8}{15}\pi^4 - 48 \sum_{k=1}^{N} \frac{(-1)^k}{k^4} \cos(kx)$$

for enough values of N to verify that $\lim_{N \to \infty} S_N = f_{PER}$

TWO CURIOSITIES

- Plug $x = \pi$ into $\circledast$:

$$0 = \frac{8}{15}\pi^4 - 48 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^4} (-1)^k$$

gives

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}}$$

- Plug $x = 0$ into $\circledast$:

$$\pi^4 = \frac{8}{15}\pi^4 - 48 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^4}$$

gives

$$\boxed{\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4} = \frac{-7}{720} \cdot \pi^4}$$

→ There are lots of similar sorts of proofs of 1st formula

→ This is only proof I know yet of 2nd

→ Although it is "easy" to show a series converges, having a simple ~~for~~ expression for sum is rare / impossible.

$\sum_{k=1}^{\infty} \frac{1}{k^3}$ "UNKNOWN"

DEF 4 $f : [a,b] \to \mathbb{R}$ is <u>PIECEWISE CTS</u> (PW CTS)
if

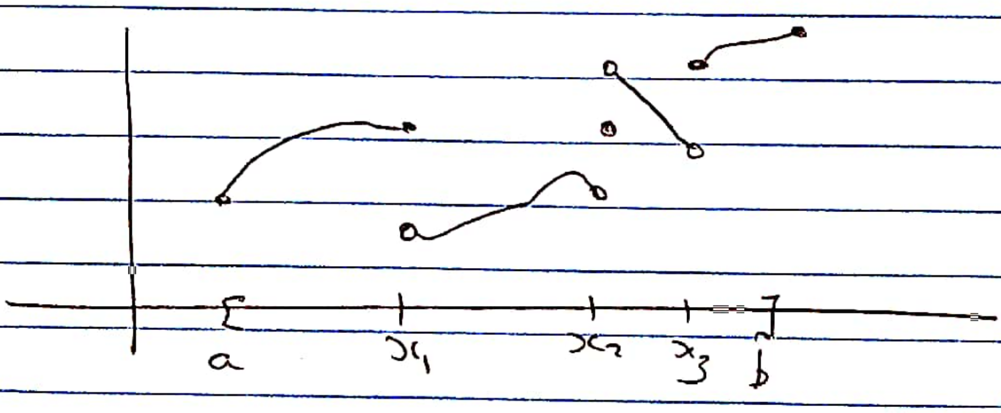
(a) f is defined + CTS on $[a,b]$ except possibly at a finite # of points

$$a \le x_1 < x_2 < \dots < x_n \le b$$

(b) At each discontinuity x_k we require

$$\lim_{x \to x_k^-} f(x) =: f(x_k^-) \quad \exists$$

$$\lim_{x \to x_k^+} f(x) =: f(x_k^+) \quad \exists$$



We say $f : \mathbb{R} \to \mathbb{R}$ is PW CTS if

$\forall\ [a,b] \subseteq \mathbb{R}$, $f : [a,b] \to \mathbb{R}$ is PW CTS.

DEF 5 $f : [a,b] \to \mathbb{R}$ is **PIECEWISE C^1** if

ⓐ f is C^1 on $[a,b]$ except possibly at a finite # of pts

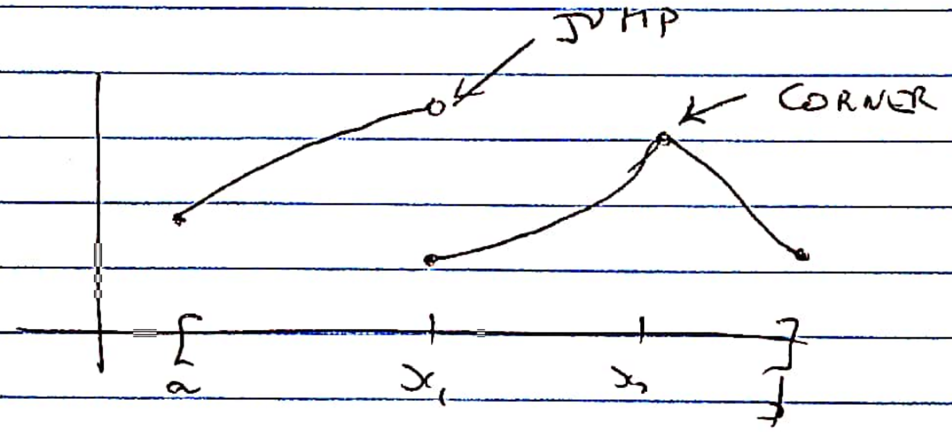
$$a \le x_1 < \dots < x_n \le b.$$

ⓑ At each x_k

$$\lim_{x \to x_k^\pm} f(x) =: f(x_k^\pm) \ \exists$$

$$\lim_{x \to x_k^\pm} f'(x) =: f'(x_k^\pm) \ \exists.$$

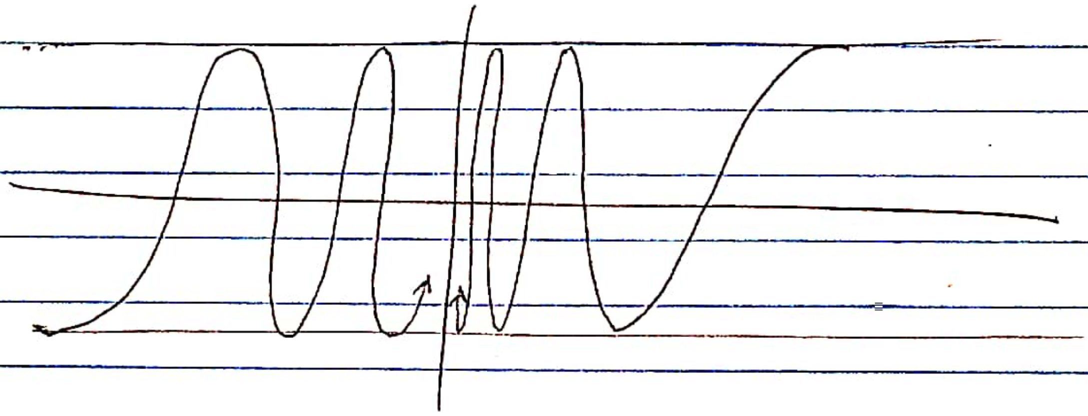
EX



Ex 6

(a) $f(x) = \sin\left(\frac{1}{x}\right)$ on $[-1, 1]$ is NOT PW CTS

as $\lim\limits_{x \to 0^{\pm}} \sin \frac{1}{x}$ DNE

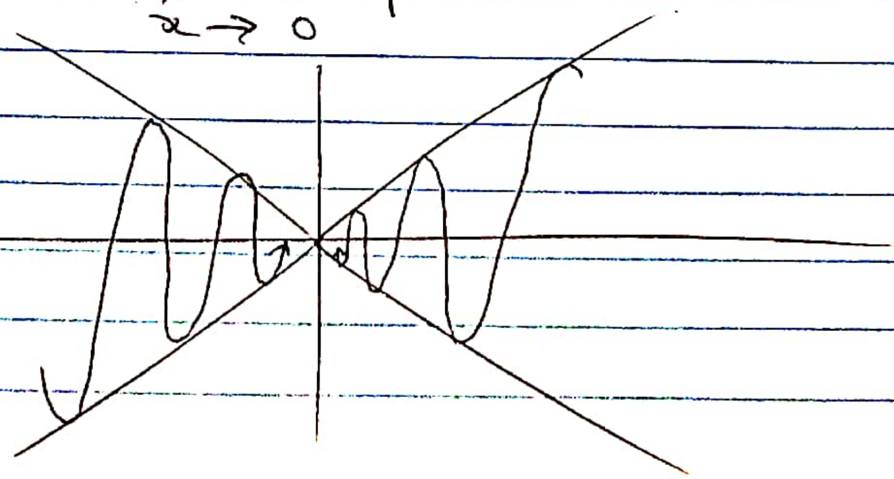


(b) $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{IF } x \neq 0 \\ 0 & \text{IF } x = 0 \end{cases}$

is CTS but NOT PW C^1 on $[-1, 1]$

as $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x} \cos x \qquad x \neq 0$

and so $\lim\limits_{x \to 0} f'(x)$ DNE.



PERIODIC EXTENSIONS

(A) F.S. of f

Start with $f : [-\pi, \pi] \longrightarrow \mathbb{R}$

Obtain

$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx$$

$$S_N : \mathbb{R} \longrightarrow \mathbb{R}$$

Clearly $S_N(x + 2\pi) = S_N(x)$ so 2π-periodic

If

$$\tilde{f}(x) = \lim_{n \to \infty} S_N(x) \quad \exists \ \forall x \in \mathbb{R}$$

Then

$$\tilde{f}(x + 2\pi) = \hat{f}(x) \quad \text{MUST HOLD}$$

(B) PERIODIC EXTENSION of f

Given $f : [-\pi, \pi) \longrightarrow \mathbb{R}$ define

$f_{PER} : \mathbb{R} \longrightarrow \mathbb{R}$ ~~to~~ be 2π-periodic so that

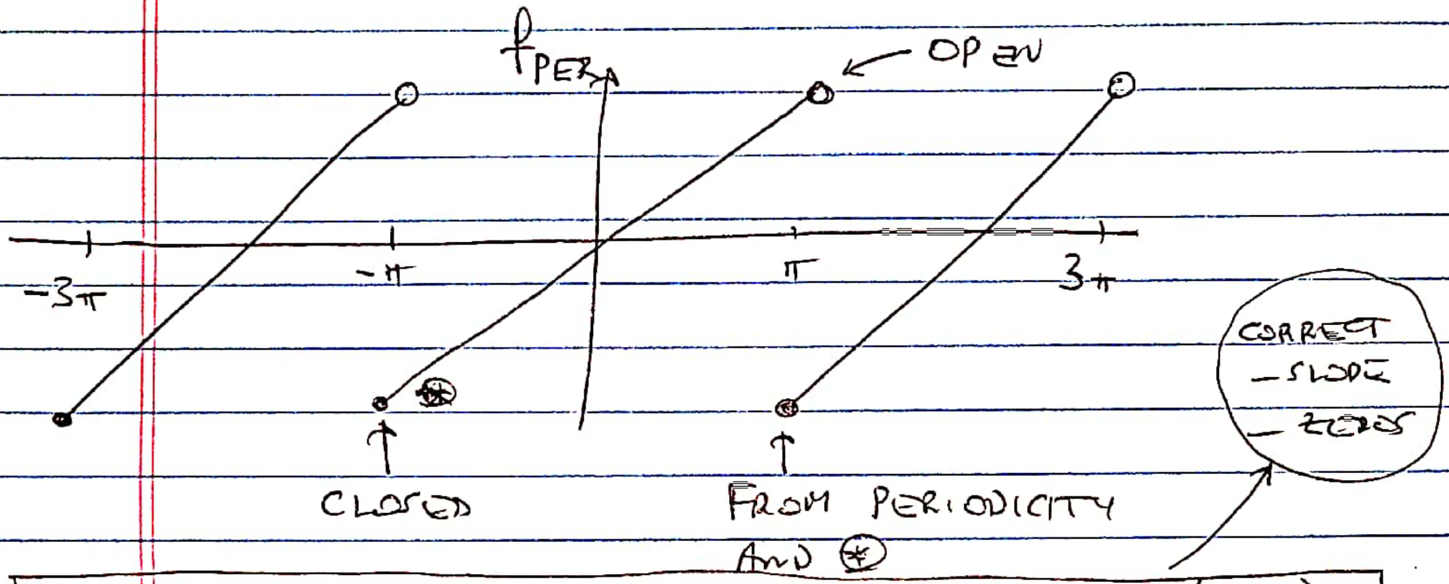
$$f_{PER}(x) = f(x) \quad \forall x \in [-\pi, \pi]$$

EX See EX 2 $f(x) = (x^2 - \pi^2)^2$ above.

MAYBE $\boxed{f_{PER} = \tilde{f}}$ IF f is NICE ENOUGH?

Ex 7

1. $f(x) = x$ on $[-\pi, \pi)$



$$\boxed{f_{PER}(x) = x - 2m\pi \quad \text{for} \quad (2m-1)\pi \leq x < (2m+1)\pi}$$

2. $f(x) = \text{sign}(x)$ on $[-\pi, \pi)$