# FOURIER SERIES I [OLVER, 3.2]

**DEF 1** Let $C^0([-\pi,\pi]) = \{ f : [-\pi,\pi] \xrightarrow{\text{cts}} \mathbb{R} \}$ be the vector space of cts functions on $[-\pi,\pi]$. Endow $C^0$ with the inner product

$$\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) g(x)\, dx$$

**CHECK AXIOMS**

(a) $\langle f, f \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx \geq 0$

THM If $g \geq 0$ is cts and $\int_a^b g(x)\, dx = 0$ Then $g \equiv 0$.

So $\langle f, f \rangle = 0 \Leftrightarrow f \equiv 0$

(b) $\langle \alpha f_1 + f_2, g \rangle = \alpha \langle f_1, g \rangle + \langle f_2, g \rangle$ ✓

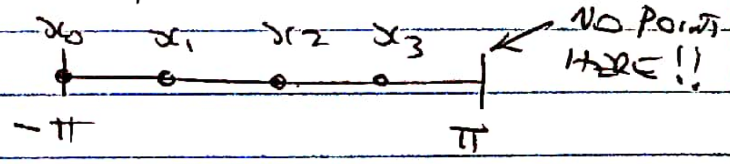
(c) $\langle f, g \rangle = \langle g, f \rangle$ ✓

**DEF 2** $\|f\| := \sqrt{\langle f, f \rangle} = \frac{1}{\sqrt{\pi}} \sqrt{\int_{-\pi}^{\pi} |f(x)|^2 dx}$ is the $L^2$-NORM on $C^0([-\pi,\pi])$.

## CONNECTION TO STANDARD INNER PRODUCT ON $\mathbb{R}^M$

Let $x_k = -\pi + k\Delta x, \qquad k = 0, 1, 2, \ldots, N-1$

where $\Delta x = \dfrac{2\pi}{M}$. $\qquad M = 4$




Let $f_k = f(x_k)$.

FORM

$$\vec{f} = \begin{pmatrix} f_0 \\ \vdots \\ f_{M-1} \end{pmatrix} \in \mathbb{R}^M$$

Then

$$\langle f, g \rangle_{C^0} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

$$\simeq \frac{1}{\pi}\sum_{k=0}^{M-1} f(x_k)\, g(x_k)\, \Delta x \qquad \text{RIEMANN SUM}$$

$$\boxed{\langle f, g \rangle_{C_0} \simeq \frac{2}{M} \langle \vec{f}, \vec{g} \rangle_{\mathbb{R}^M}}$$

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**DEF 3** $\tilde{c}_0(x) = \frac{1}{\sqrt{2}}$

$c_k(x) = \cos(kx)$

$k = 1, 2, 3, \ldots$

$s_k(x) = \sin(kx)$

For each $N$

$$F_N = \operatorname{Span}\{\tilde{c}_0, c_1, \ldots, c_N, s_1, \ldots, s_N\} = \operatorname{Span}(F_N)$$
$$\subseteq C^0([-\pi, \pi])$$

**CLAIM 4** (1) $F_N$ is an ORTHONORMAL SET IN $C^0$.

(2) Hence $F_N$ is an ON BASIS for $F_N$

and $\dim(F_N) = 2N+1$

**PROOF**

(1) $\|\tilde{c}_0\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\,dx = 1$

$k \geq 1$ $\|c_k\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(kx)\,dx$

$$\overset{\theta = kx}{=} \frac{1}{k\pi}\int_{-k\pi}^{k\pi} \cos^2(\theta)\,d\theta = \frac{1}{k\pi}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{-k\pi}^{k\pi}$$
$$= 1$$

SIMILARLY $\|s_k\|^2 = 1$

ALSO $\langle c_k, s_\ell \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underset{\text{EVEN}}{\cos(kx)} \times \underset{\text{ODD}}{\sin(\ell x)}}_{\text{ODD}}\, dx = 0$

and for $\ell \neq k$

$$\langle c_k, c_\ell \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(kx)\cos(\ell x)\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos[(k+\ell)x] + \cos[(k-\ell)x]\, dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(k+\ell)x}{k+\ell} + \frac{\sin(k-\ell)x}{k-\ell}\right]_{-\pi}^{\pi} = 0$$

Similarly $\langle s_k, s_\ell \rangle = 0$ for $\ell \neq k$

$\square$

CLAIM 5 $\dim C^0 = \infty$

PF $\forall N$, $F_N$ is a LI set with $2N+1$ elts

So $\nexists$ maximal LI set in $C^0$ $\square$

OBSERVATION

LET $f \in F_N$. Since we have ONB

$$f(x) = \tilde{a}_0 \tilde{c}_0 + \sum_{k=1}^{N} a_k e_k(x) + b_k s_k(x)$$

with $\quad a_k = \langle c_k, f \rangle$, $\quad b_k = \langle s_k, f \rangle$ and

$$\tilde{a}_0 = \langle \tilde{c}_0, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f(x)\, dx$$

SET

$$\boxed{a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \text{AVERAGE VALUE OF } f \text{ ON } [-\pi, \pi]}$$

Then

$$\hat{a}_0 = \frac{1}{\sqrt{2}} a_0, \qquad \hat{c}_0 = \frac{1}{\sqrt{2}}$$

So $\quad \tilde{a}_0 \tilde{c}_0 = \frac{1}{2} a_0.$

DEF 6 Let $f : [-\pi, \pi] \to \mathbb{R}$. The FOURIER SERIES of $f$

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \quad (*)$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx \qquad k = 0, 1, 2, \ldots$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx \qquad k = 1, 2, \ldots$$

NOTES

1. There is NO GAURANTEE (YET) that the series on RHS of (*) converges for an $x$

2. If series does converge at $x$
There is no gaurantee that the sum of series equals $f(x)$.

Hence notation $\sim$ instead of $=$

3. Suppose that series on RHS converges to $g(x)$ for each $x \in [-\pi, \pi]$.

We don't (yet) know if $g$ is CTS or if $g$ is differentiable, even when $f$ is.

EXS

1 [See Olver p 75]

Let $f(x) = x$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \, dx = 0$$

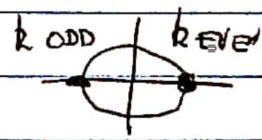
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x}_{\text{ODD}} \underbrace{\cos(kx)}_{\text{EVEN}} \, dx = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(kx) \, dx$$

$$\overset{\text{PARTS}}{=} \frac{1}{\pi}\left[ -\frac{x\cos(kx)}{k} + \frac{\sin(kx)}{k^2} \right]_{x=-\pi}^{x=\pi}$$

$$= -\frac{\cos(k\pi)}{k} \mp \frac{\cos(-k\pi)}{k}$$

$$= \frac{-2}{k}\cos(k\pi) = \frac{-2}{k}(-1)^k$$




$$\boxed{b_k = \frac{2}{k}(-1)^{k+1}.}$$

BE SUPER CAREFUL WHEN DOING CALCULATIONS LIKE THIS!

UPSHOT

$$x \sim 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\sin(kx) \qquad (\dagger)$$

$$= 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\right)$$

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NOTE

1. If you fix $x$, RHS of (†) is a series of real numbers.

For most values of $x$, the standard tests for convergence of series from Calc II are INCONCLUSIVE

ISSUES
- Because $\sin(kx)$ can take both $+, -$ values as $k \uparrow$, series is not alternating
- $$\sum_{k=1}^{\infty} \left| \frac{\sin(kx)}{k} \right| \leq \sum_{k=1}^{\infty} \frac{1}{k}$$

  is obvious estimate, but RHS DIVERGES.

2. Later we will show (†) does converge $\forall x$

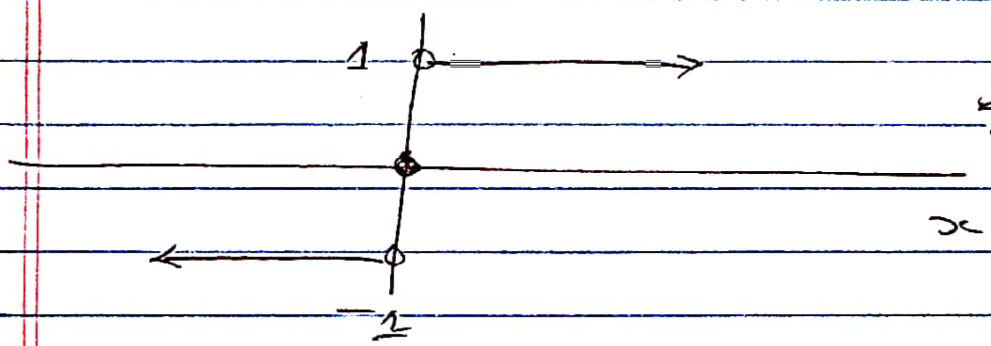
But what $f^n$ does it converge?

It CANNOT agree with $f(x) = x$ everywhere

EX At $x = \pi$, Series is $0$ but $f(\pi) = \pi$.

EXS
2

$$f(x) = \text{sign}(x) = \begin{cases} 1 & \text{IF } x > 0 \\ 0 & \text{IF } x = 0 \\ -1 & \text{IF } x < 0 \end{cases}$$



$f$ IS ODD

For $k \geq 0$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\text{sign}(x)}_{\text{ODD}} \underbrace{\cos(kx)}_{\text{EVEN}} \, dx = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{sign}(x) \sin(kx) \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} -\sin(kx) \, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin(kx) \, dx$$

$$\overset{\sin \text{ ODD}}{=} \frac{2}{\pi} \int_{0}^{\pi} \sin(kx) \, dx$$

$$= \frac{2}{\pi} \left[ -\frac{\cos(kx)}{k} \right]_{0}^{x=\pi}$$

$$= \frac{2}{\pi} \left[ \frac{1}{k} - \frac{\cos k\pi}{k} \right]$$

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$$= \frac{2}{\pi}\left[\frac{1}{k} - \frac{(-1)^k}{k}\right]$$

$$= \frac{2}{\pi k}\left[1 + (-1)^{k+1}\right] = \begin{cases} \frac{4}{\pi k} & k \text{ ODD} \\ 0 & k \text{ EVEN} \end{cases}$$

So writing $k = 2n+1$, $n = 0, 1, 2, \ldots$

NOTE

$$\boxed{\operatorname{sign}(x) \sim \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin[(2n+1)x]}$$

CARE REQUIRED ONCE AGAIN!

- We have same issues re convergence as in last example!