# 7.3, 7.4 MATRIX EXPONENTIATION + ODE SYSTEMS

Recall

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \qquad z \in \mathbb{C}$$

If $A$ is $n \times n$ we could formally define

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \qquad \text{AN } n \times n \text{ MATRIX} \quad (1)$$

assuming $\infty$-series converged. (It does).

How can we calculate $e^A$ in practice?

If $A$ is diagonalizable $A = PDP^{-1}$

with $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ then

$$A^k = P D^k P^{-1} \qquad \text{with} \quad D^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$$

So

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} P D^k P^{-1} = P \left( \sum_{k=0}^{\infty} \frac{1}{k!} D^k \right) P^{-1}$$

$$= P \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_1^k & & \\ & \ddots & \\ & & \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_n^k \end{bmatrix} P^{-1}$$

$$= P \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} P^{-1}$$

<u>SUMMARY</u> Suppose $A = PDP^{-1}$! Define

$$e^D = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} \quad \text{Then} \quad \boxed{e^A = P e^D P^{-1}} \quad (2)$$

**EX A** Calculate $e^A$ for your favorite $3 \times 3$ matrix $A$ that is diagonalizable but for which one eigenvalue has multiplicity 2

**ISSUE** $P$ may not be unique. So how do we know $e^A$ given by (2) is uniquely defined?

**SOLUTION** [CASE A IS NORMAL] Use Spectral Decompos$^n$.

$$e^A = P e^D P^{-1} = [X_1 | \cdots | X_k] \cdot \begin{bmatrix} e^{\lambda_1 I} & & \\ & \ddots & \\ & & e^{\lambda_k I} \end{bmatrix} \begin{bmatrix} X_1^* \\ \vdots \\ X_k^* \end{bmatrix}$$

$$= e^{\lambda_1} X_1 X_1^* + \cdots + e^{\lambda_k} X_k X_k^*$$

$$\boxed{e^A = e^{\lambda_1} G_1 + \cdots + e^{\lambda_k} G_k}$$

# MORE GENERALLY

Suppose $f : \mathbb{C} \to \mathbb{C}$ is any function and $A = PDP^{-1}$.

We define

$$f(A) = P\, f(D)\, P^{-1}$$

$$f(A) = f(\lambda_1)\, G_1 + \cdots + f(\lambda_k)\, G_k$$

EX $f(z) = \sin z$
$f(z) = \cos z$

RECALL $G_i = \prod_{\substack{j=1 \\ j \neq i}}^{k} (A - \lambda_j I) \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{k} (\lambda_i - \lambda_j)$

is a polynomial of degree $\leq k-1$ in $A$.

So $f(A)$ is ~~too~~ (even though $f(z) = e^z$ or $f(z) = \sin z$ has $\infty$ power series in $z$)

CAVEAT $\lambda_j = \lambda_j(A)$ depends on $A$ ~~too~~.

**APPLICATION** Solve IVP

$$\begin{cases} \dfrac{d\vec{u}}{dt} = A\vec{u} \\ \vec{u}(0) = \vec{u}_0 \end{cases}$$

A diagonalizable

**CLAIM.**

$$\vec{u}(t) = e^{\lambda_1 t}\vec{v}_1 + \dots + e^{\lambda_k t}\vec{v}_k$$

where $\vec{v}_j := f_j \vec{u}_0$ is an evector of $A$ with evalue $\lambda_j$.

(Recall $f_j$ projects onto $N(A - \lambda_j I)$)

PF

(1) Since we have

$$e^{At} = e^{\lambda_1 t} f_1 + \dots + e^{\lambda_k t} f_k$$

$$\frac{d}{dt}(e^{At}) = \lambda_1 e^{\lambda_1 t} f_1 + \dots + \lambda_k e^{\lambda_k t} f_k$$

$$= (\lambda_1 f_1 + \dots + \lambda_k f_k)(e^{\lambda_1 t} f_1 + \dots + e^{\lambda_k t} f_k)$$

as $f_i f_j = \delta_{ij} f_i$

$$= A e^{At}$$

$$= e^{At} A$$

And by similar arguements

$$e^{-At}e^{At} = I = e^{0}$$

2 Let $\vec{u}(t) = e^{At}\vec{u}_0$.

Then $\dfrac{d\vec{u}}{dt} = Ae^{At}\vec{u}_0$ by 1
$= A\vec{u}$.

and $\vec{u}(0) = e^{0}\vec{u}_0 = \vec{u}_0$. D.

$\boxed{\text{EXERCISE B}}$ For matrix $A$ you picked in EX A

solve $\begin{cases} \dfrac{d\vec{u}}{dt} = A\vec{u} \\ \vec{u}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \end{cases}$