APPENDIX

COMPLETION OF PROOF OF S.T. FOR NORMAL MATRICES.

$\boxed{③ \Rightarrow ②}$ Let A be normal with spectrum

$$\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}.$$

You can check that $A - \lambda_k I$ is also normal.

So by the Range-Nullspace Decomposition (RND) Theorem for Normal matrices (see P 408 and 409 / Last Lecture)

$\exists$ Unitary matrix U_k and invertible matrix C_k with

$$U_k^* (A - \lambda_k I) U_k = \left(\begin{array}{c|c} C_k & 0 \\ \hline 0 & 0 \end{array}\right)$$

where (2,2) block is a square matrix of size

$$\dim N(A - \lambda_k I) \equiv \text{Geometric Multiplicity of } \lambda_k =: a_k$$

So

$$U_k^* A U_k - \lambda_k U_k^* U_k = \left(\begin{array}{c|c} C_k & 0 \\ \hline 0 & 0 \end{array}\right)$$

or

$$U_k^* A U_k = \left(\begin{array}{c|c} C_k + \lambda_k I & 0 \\ \hline 0 & \lambda_k I_{a_k} \end{array}\right) =: \begin{pmatrix} A_{k-1} & 0 \\ 0 & \lambda_k I_{a_k} \end{pmatrix}$$

GOOD: STARTING TO LOOK DIAGONAL!

CLAIM I

① $\lambda_k \notin \sigma(A_{k-1})$

Hence

② $a_k = \text{Alg Mult}(\lambda_k)$.

PF If $\lambda_k \in \sigma(A_{k-1})$, then $C_k = A_{k-1} - \lambda_k I$ would have 0 as an eigenvalue, ie C_k would not be invertible, which is a contradiction to RND Theorem above.

CLAIM II

A_{k-1} is Normal.

PF You do it.

So we can repeat above to get

$$U_{k-1}^* A_{k-1} U_{k-1} = \left(\begin{array}{c|c} A_{k-2} & 0 \\ \hline 0 & \lambda_{k-1} I_{a_{k-1}} \end{array}\right)$$

Let

$$\tilde{U}_j = \left(\begin{array}{c|c} U_j & 0 \\ \hline 0 & I_{a_{j+1}+\cdots+a_k} \end{array}\right)$$

and

$$U = \tilde{U}_k \tilde{U}_{k-1} \cdots \tilde{U}_2 \tilde{U}_1 .$$

You can check that

$$U^* A U = \begin{pmatrix} \lambda_1 I_{a_1} & & & 0 \\ & \boxed{\lambda_2 I_{a_2}} & & \\ & & \ddots & \\ 0 & & & \lambda_k I_{a_k} \end{pmatrix} = D$$

is diagonal

OR $\quad A = U D U^* \quad$ as required $\square$

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