7.5 SPECTRAL THM FOR NORMAL MATRICES

A is $n \times n$ real or complex matrix throughout.

Recall

SPECTRAL THM

A has basis of evectors of $\mathbb{C}^n$ $\Leftrightarrow$ $A = PDP^{-1}$ is diagonalizable

QN When does a diagonalizable matrix have an ONB of evectors?

SCAFFOLDED PROBLEM A Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

1. Show $\lambda_1 = 1$, $\lambda_2 = 2$
2. Show $N(A - \lambda_1 I) = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

   $N(A - \lambda_2 I) = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$
3. Explain why ~~A does not have~~ $\nexists$ basis $\{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{C}^2$ so that

   a. $\vec{v}_1, \vec{v}_2$ are evectors of A

   b. $\vec{v}_1 \perp \vec{v}_2$.

SPECTRAL THM FOR NORMAL MATRICES

TFAE

1. A has ONB of eigenvectors
2. $A = UDU^*$ where D is diagonal and U is unitary $(U^* = U^{-1})$
3. A is Normal, ie $AA^* = A^*A$

PF

$\boxed{1 \Leftrightarrow 2}$

Same as proof of original Spectral Thm

+ fact

U unitary $\Leftrightarrow$ Cols of U form ONB

$\boxed{2 \Rightarrow 3}$

$$
\begin{aligned}
AA^* &= (UDU^*)((UDU^*)^* \\
&= UDU^*UD^*U^* \\
&= UDD^*U \\
&= UD^*DU \quad \text{as Diagonal matrices commute} \\
&= \dots = A^*A
\end{aligned}
$$

$\boxed{3 \Rightarrow 2}$

See ~~end of lecture~~ textbook.

LEMMA 1

Let A be normal and λ_1, λ_2 be distinct evalues of A

Then

$$N(A - \lambda_2 I) \perp N(A - \lambda_1 I)$$

PF

We will need the following

CLAIM [Meyer p 212 eqn (4.5.6)]

For any matrix B

$$N(B^* B) = N(B)$$

Suppose $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$ with $\lambda_2 \neq \lambda_1$

Then

$$N(A - \lambda_1 I) \overset{\text{CLAIM}}{=} N[(A - \lambda_1 I)^*(A - \lambda_1 I)]$$

$$= N[(A - \lambda_1 I)(A - \lambda_1 I)^*]$$

as A and hence $A - \lambda_1 I$ is normal

$$\overset{\text{CLAIM}}{=} N((A - \lambda_1 I)^*)$$

So we know $(A-\lambda_1 I)^* \vec{v}_1 = \vec{0}$

So

$$\vec{v}_1^{\,*}(A-\lambda_1 I) = \vec{0}$$

$$\Rightarrow \quad \vec{v}_1^{\,*}(A-\lambda_1 I)\vec{v}_2 = \vec{0}$$

$$\Rightarrow \quad \vec{v}_1^{\,*}(\lambda_2-\lambda_1)\vec{v}_2 = \vec{0} \quad \text{as} \quad A\vec{v}_2 = \lambda_2 \vec{v}_2$$

$$\Rightarrow \quad \vec{v}_1^{\,*}\vec{v}_2 = \vec{0} \quad \text{as} \quad \lambda_2 \neq \lambda_1$$

$$\Rightarrow \quad \vec{v}_1 \perp \vec{v}_2$$

$\square$

LEMMA 2

① If $A^* = A$ (A is real symmetric or Hermitian) Then all evalues of A are real

② If $A^T = -A$ and $A \in \mathbb{R}^{n\times n}$ Then all evalues of A are pure imaginary

EX $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has $\sigma(A) = \{\pm i\}$

PF ① IF $A\vec{v} = \lambda \vec{v}$ WITH $\vec{v} \neq \vec{0}$ THEN

$$\lambda \|\vec{v}\|^2 = \lambda \langle \vec{v} | \vec{v} \rangle = \langle \vec{v} | \lambda \vec{v} \rangle = \langle \vec{v} | A\vec{v} \rangle$$
$$= \langle A^*\vec{v} | \vec{v} \rangle = \langle A\vec{v} | \vec{v} \rangle = \langle \lambda \vec{v} | \vec{v} \rangle = \bar{\lambda} \langle \vec{v} | \vec{v} \rangle = \bar{\lambda} \|\vec{v}\|^2$$

So $(\lambda - \bar{\lambda}) \|\vec{v}\|^2 = 0$

Since $\vec{v} \neq \vec{0}$, $\lambda = \bar{\lambda}$. So $\lambda \in \mathbb{R}$

② SIMILAR. YOU DO IT!

SPECTRAL THM FOR REAL SYMMETRIC MATRICES

Let $A \in \mathbb{R}^{n \times n}$, $A^T = A$

Then $A = PDP^T$ where D is diagonal and P is orthogonal.

SCAFFOLDED PROBLEM ⓑ

THIS PROBLEM SHOWS YOU HOW TO OBTAIN S.T. FOR SYMMETRIC MATRICES FROM S.T. FOR NORMAL MATRICES

Let $A \in \mathbb{R}^{3\times 3}$, $A^T = A$ be

$$A = \begin{pmatrix} 7 & -1 & -2 \\ -1 & 7 & 2 \\ -2 & 2 & 10 \end{pmatrix}$$

① SHOW $p(\lambda) = \det(A - \lambda I) = -(\lambda - 6)^2 (\lambda - 12)$

② Use the S.T. for Normal Matrices ~~to~~ explain why $\dim N(A - 6I) = 2$ must hold.

③ Now calculate bases for $N(A - 6I)$, $N(A - 12I)$ and verify that

$$N(A - 6I) \perp N(A - 12I)$$ holds.

④ For a general real symmetric matrix, A, explain why we can always obtain bases for each of the eigenspaces of A that consist of real (rather than complex) vectors.

⑤ Use fact that $\lambda = 6$ is real together with Gram - Schmidt to calculate an ONB for $N(A - 6I)$ consisting of real vectors.

⑥ Suppose λ is an eigenvalue of a real symmetric matrix, A, with $AlgMult(\lambda) > 1$. Explain why the method in ⑤ can be used to obtain an ONB of real vectors for $N(A - \lambda I)$. Why is $\dim N(A - \lambda I) = AlgMult(\lambda)$?

⑦ For the 3x3 matrix, A, ~~in ①~~ above calculate an ONB for $\mathbb{R}^3$ consisting of eigenvectors of A (HINT: Use ③ and ⑥)

⑧ Finally show $A = PDP^T$ where $D = \mathrm{Diag}(6, 6, 12)$ and P is orthogonal matrix

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{3} & 1/\sqrt{6} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{pmatrix}$$

⑨ Generalize these calculations to prove S.T. for Real Symmetric matrices.

2ND DERIVATIVE TEST (CALCULUS 3)

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $z = f(\vec{x})$. Suppose $\vec{x}_0$ is a critical point of f, ie $\nabla f(\vec{x}_0) = \vec{0}$.

Let

$$\Delta = \det(H) \quad \text{where} \quad H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\Bigg|_{\vec{x}=\vec{x}_0} \qquad \underline{\text{HESSIAN}}$$

Then

Δ	$f_{xx}(\vec{x}_0)$	CLASSIFICATION
+	+	LOCAL MIN
+	−	LOCAL MAX
−	*	SADDLE POINT

<u>PF</u> By Taylor's series

$$z = f(\vec{x}) \simeq f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) + (\vec{x} - \vec{x}_0)^T H (\vec{x} - \vec{x}_0)$$

Let $z' = z - f(\vec{x}_0)$, $\vec{x}' = \vec{x} - \vec{x}_0$. Since $\nabla f(\vec{x}_0) = \vec{0}$ we just need to classify. $Q(\vec{x}) = \vec{x}^T H \vec{x}$ at $\vec{x} = \vec{0}$. Since H is real symmetric, $H = P^T D P$ where can chose $\det(P) = +1$. So P effects a rotation, $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some θ

Since rotating coords does not change nature of CPT we can look at $\tilde{Q}(\vec{y}) := Q(P^T\vec{y}) = Q(\vec{x}) = \vec{x}^T P^T D P \vec{x}$

$= \vec{y}^T D \vec{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2$.

$\Delta = \det(H) = \det(D) = \lambda_1 \lambda_2$.

$f_{xx} = (P^T D P)_{11} = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$

Sign $(\lambda_1 \lambda_2)$	Sign (λ_1)	CLASS^n
+	+	MIN
+	−	MAX
−	*	SADDLE

ISSUE Suppose choose $D = \mathrm{Diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and $\exists$ repeated evalues.
Then $\exists\ \infty$ many choices for P, ie $\boxed{P \text{ is NOT UNIQUE.}}$

For this reason it can be hard to define e^A.

BUT we have

THE SPECTRAL DECOMPOSITION OF A NORMAL MATRIX

Let A be $n\times n$ normal with $\sigma(A) = \{\lambda_1, \dots, \lambda_k\}$.

So $A = UDU^*$ where $U = [X_1 | X_2 | \dots | X_k]$

and X_j is $n \times a_j$ matrix whose cols are

ONB for $N(A - \lambda_j I)$.

So
$$A = [X_1 | \dots | X_k] \begin{bmatrix} \lambda_1 I_{a_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_k I_{a_k} \end{bmatrix} \begin{bmatrix} X_1^* \\ \vdots \\ X_k^* \end{bmatrix}$$

ie
$$\boxed{A = \lambda_1 X_1 X_1^* + \dots + \lambda_k X_k X_k^*}$$

SPECIAL CASE If all evalues have alg mult $= 1$
Then
$$A = \lambda_1 \vec{v}_1 \vec{v}_1^* + \dots + \lambda_k \vec{v}_k \vec{v}_k^* \qquad A\vec{v}_j = \lambda_j \vec{v}_j$$
where $\{\vec{v}_1, \dots, \vec{v}_n\}$ is ONB for $\mathbb{C}^n$ of evectors.

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$A\vec{v}_j = \lambda_j \vec{v}_j$

where $\{\vec{v}_1, \dots, \vec{v}_n\}$ is ONB for $\mathbb{C}^n$ of evectors.

RECALL Let $\|\vec{v}\| = 1$. Then $P(\vec{w}) := \vec{v}\,\vec{v}^*\vec{w} = \langle \vec{v} | \vec{w} \rangle \vec{v}$ is $\perp$ projection onto $\mathrm{Span}(\vec{v})$.

Since $\mathrm{Span}\{-\vec{v}\} = \mathrm{Span}\{\vec{v}\}$ there are 2 possible choices for $\vec{v}$. However the L.T. P is the same for both.

GENERAL CASE : GEOMETRIC MEANING of $X_j X_j^*$

Since $A - \lambda_j I$ is NORMAL, the Range–Nullspace Decomposition tells us

$$\mathbb{C}^n = N(A - \lambda_j I) \oplus R(A - \lambda_j I)$$

where $N(A - \lambda_j I)^{\perp} = R(A - \lambda_j I)$

Let $G_j =$ Orthogonal Projector onto $N(A - \lambda_j I)$ along $R(A - \lambda_j I)$.

The L.T. G_j is well-defined indept of choice of basis for these ~~two~~ complementary spaces.

By 5.9, 5.11 we get

$$E_j = [X_j \mid 0]\,[X_j \mid Y_j]^{-1}$$

ONB's for N, R

$$= [X_j \mid 0] \begin{bmatrix} X_j^* \\ Y_j^* \end{bmatrix}$$

$$= X_j X_j^*$$

SPECTRAL DECOMPN THM

Let A be Normal, E_j as above. Then

① $A = \lambda_1 E_1 + \cdots + \lambda_k E_k$ SPECTRAL DECOMPOSITION OF A

② $I = E_1 + \cdots + E_k$

③ $E_i E_j = 0$ if $i \neq j$

NOTE A generalization of this Thm holds for ANY $n \times n$ MATRIX !!

Pf

1. See above

2. $I = UU^* = [X_1 \dots X_k] \begin{bmatrix} X_1^* \\ \vdots \\ X_k^* \end{bmatrix} = X_1 X_1^* + \dots + X_k X_k^*$
$= \sigma_1 + \dots + \sigma_k$

3. $\sigma_1 \sigma_2 = X_1 \underbrace{X_1^* X_2}_{=0} X_2^* = 0$

since by Lemma 1 if $A\vec{v}_j = \lambda_j \vec{v}_j \quad j = 1, 2$

Then $\vec{v}_1 \perp \vec{v}_2$ So $v_1^* \vec{v}_2 = 0$.

SCAFFOLDED PROBLEM C

1. Compute the Spectral Decomposition for the matrix $A = UDU^*$ where

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

SLIGHTLY DIFFERENT FROM S.P. B

(2) You should find that for $\lambda_1 = 2$

we have $G_1 = X_1 X_1^*$ where X_1 is 3×1

and for $\lambda_2 = 3$, $G_2 = X_2 X_2^*$ where X_2 is 3×2

Calculate the matrices G_1 and G_2

(3) Check that

$$G_1 + G_2 = I$$
$$G_1 G_2 = 0$$

and

$$A = 2G_1 + 3G_2.$$

MAGIC FORMULA FOR G_j

$\boxed{\text{CASE } k=3}$

Suppose $\sigma(A) = \{\lambda_1, \lambda_2, \lambda_3\}$

So $A = \lambda_1 G_1 + \lambda_2 G_2 + \lambda_3 G_3$ (1)

$I = G_1 + G_2 + G_3$ (2)

AS A PROJECTOR

$G_i G_j = 0 \quad i \neq j, \qquad G_j^2 = G_j$ (3)

Let solve for G_j!

By 1 and 3

$$A - \lambda_1 I = (\lambda_2 - \lambda_1) G_2 + (\lambda_3 - \lambda_1) G_3$$

$$A - \lambda_2 I = (\lambda_1 - \lambda_2) G_1 + (\lambda_3 - \lambda_2) G_3$$

So

$$(A - \lambda_1 I)(A - \lambda_2 I)$$

$$= [(\lambda_2 - \lambda_1) G_2 + (\lambda_3 - \lambda_1) G_3][(\lambda_1 - \lambda_2) G_1 + (\lambda_3 - \lambda_2) G_3]$$

$$= \underbrace{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}_{\text{SCALARS!}} G_3 \qquad \text{by } 3$$

So

$$\boxed{G_3 = \frac{(A - \lambda_1 I)(A - \lambda_2 I)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}}$$

SCAFFOLDED PROBLEM D

1. What are G_1, G_2 when $k = 2$?
2. Use 1 to rederive formula in SP C.
3. Show in general

$$G_i = \prod_{\substack{j=1 \\ j \neq i}}^{k} (A - \lambda_j I) \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{k} (\lambda_i - \lambda_j).$$