7.2 DIAGONALIZATION

(A) TODAY: $A \in \mathbb{C}^{n \times n}$ or $A \in \mathbb{R}^{n \times n}$.

DEF 1 (a) A COMPLETE SET OF EVECTORS FOR A is any set of n LI evectors of A

(b). A is DIAGONALIZABLE if $\exists$ $n \times n$ P, D:

$$A = PDP^{-1} \quad \text{where } D \text{ IS DIAGONAL}$$

SPECTRAL THM I

A DIAGONALIZABLE $\Leftrightarrow$ A HAS COMPLETE SET OF EVECTORS.

In this case, if $(\lambda_1, \vec{v}_1), \ldots, (\lambda_n, \vec{v}_n)$ are a complete set of epairs for A, then

$$D = \text{Diag}[\lambda_1, \ldots \lambda_n]$$

$$P = [\vec{v}_1, \ldots, \vec{v}_n]$$ (COLS OF P are evectors)

EXERCISE FOR YOU:

EXPLAIN why $n \times n$ matrix P is invertible $\Leftrightarrow$ Cols of P are L.I.

PF OF THM

OBSERVE

① $(AP)_{*j} = A P_{*j} = A \vec{v}_j$

② $(PD)_{ij} = \sum_k P_{ik} D_{kj} = \lambda_j P_{ij}$

OR $(PD)_{*j} = \lambda_j P_{*j} = \lambda_j \vec{v}_j$

Then

$$P^{-1}AP = D \iff AP = PD$$
$$\iff (AP)_{*j} = (PD)_{*j} \quad \forall j$$
$$\iff A\vec{v}_j = \lambda_j \vec{v}_j \quad \forall j \text{ by } ①, ②$$

∎

(B) SCAFFOLDED PROBLEMS

DEF: ① The GEOMETRIC MULTIPLICITY of λ is $\dim N(A - \lambda I)$

② The ALGEBRAIC MULTIPLICITY of λ is order of root in $p(\lambda) = \det(A - \lambda I)$

EG $p(\lambda) = (\lambda - 2)^2 (\lambda - 4)$ $\quad \lambda = 2$ HAS AM 2

(I) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

ⓐ Show $\sigma(A) = \{1, 2\}$

ⓑ Show $\dim(N(A - 1I)) = 2$ and find a basis for it

ⓒ Show $\dim(N(A - 2I)) = 1$

ⓓ Diagonalize A. ⓔ Is P unique??

II SHOW $A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ with $c \neq 0$

IS NOT DIAGONALIZABLE.

ⓐ SHOW $\sigma(A) = \{1\}$

ⓑ SHOW $\dim N(A - I) = 1$

ⓒ EXPLAIN WHY A IS NOT DIAGONALIZABLE.

III SHOW $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ IS NOT DIAGONALIZABLE

ⓐ SHOW $A^3 = 0$

ⓑ Suppose $A = P^{-1} D P$
Show $D^3 = 0$

ⓒ Use fact D is diagonal to show $D = 0$

ⓓ What do you conclude about A?

ⓔ STATE A MORE GENERAL RESULT, WITH SAME PROOF.

C

SIMILARITY THM FOR EPAIRS

① IF $(\lambda, \vec{v})$ IS EPAIR FOR A

THEN $(\lambda, P\vec{v})$ IS EPAIR FOR $B = PAP^{-1}$

② IF $B = PAP^{-1}$ THEN A, B have same characteristic polynomials

③ IF $T: V \to V$ is a L.T. Then $\sigma(T)$ is well defined: PICK a basis $\mathcal{B}$ for V. ~~Let~~ Define

$$\sigma(T) = \sigma([T]_{\mathcal{B}})$$

PF ① IF $A\vec{v} = \lambda\vec{v}$ THEN

$$B(P\vec{v}) = (PAP^{-1})(P\vec{v}) = PA\vec{v} = P\lambda\vec{v} = \lambda P\vec{v}.$$

②
$$\begin{aligned} p_B(\lambda) &= \det(B - \lambda I) \\ &= \det(PAP^{-1} - \lambda I) \\ &= \det[P(A - \lambda I)P^{-1}] \\ &= \det P \cdot \det(A - \lambda I) \det(P^{-1}) \\ &= p_A(\lambda) \quad \text{as } \det(P)\det(P^{-1}) = \det(PP^{-1}) = 1 \end{aligned}$$

③ IF $\tilde{\mathcal{B}}$ is another choice of basis Then $[T]_{\tilde{\mathcal{B}}}$ is similar to $[T]_{\mathcal{B}}$. Apply ② $\square$

D) TOOLS TO DETERMINE IF A HAS COMPLETE SET OF EVECTORS

THM ON INDEPT EVECTORS

Suppose $\lambda_1, \dots, \lambda_k$ are DISTINCT evalues of A

1) IF $A\vec{v}_j = \lambda_j \vec{v}_j$ for $j = 1, \dots, k$ $\quad (\vec{v}_j \neq \vec{0})$

   THEN $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a LI set

2) IF B_j is a basis for $N(A - \lambda_j I)$

   THEN $B = B_1 \cup \dots \cup B_k$ is a LI set

3) If no evalues of A are repeated (all evalues have alg mult 1)
   Then A is diagonalizable

4) A is diagonalizable

   $$\iff \quad \text{GeoMult}_A(\lambda) = \text{AlgMult}_A(\lambda) \quad \forall \lambda \in \sigma(A)$$

THM $\forall \lambda \in \sigma(A)$

$$\text{GeoMult}_A(\lambda) \leq \text{AlgMult}_A(\lambda)$$

PF OMITTED.

Pf ① Suppose $\{\vec{v}_1 \dots \vec{v}_k\}$ are L.D. (PF by $\cdot\!\!\times\!\!\cdot$)

Order them so that

$$m = \{\vec{v}_1 \dots \vec{v}_r\} \text{ is a Max LI Set}$$

$(r < k)$

So $$\boxed{\vec{v}_{r+1} = \sum_{i=1}^{r} \alpha_i \vec{v}_i} \quad (*)$$

Then
$$0 = (A - \lambda_{r+1} I)\vec{v}_{r+1} = \sum_{i=1}^{r} \alpha_i (A - \lambda_{r+1} I)\vec{v}_i$$

$$= \sum_{i=1}^{r} \alpha_i (\lambda_i - \lambda_r)\vec{v}_i$$

Since $\{\vec{v}_1 \dots \vec{v}_r\}$ is LI

$$\alpha_i(\lambda_i - \lambda_r) = 0 \quad \forall\, i < r$$

Since $\lambda_i \neq \lambda_r$ (Evalues are distinct)

$$\alpha_i = 0 \quad \forall i$$

So by $(*)$ $\vec{v}_{r+1} = \vec{0}$ NOT POSSIBLE FOR EVECTOR

$\cdot\!\!\times\!\!\cdot$

(2) Get idea from a special case:

IF $\lambda_2 \neq \lambda_1$ and

$$N(A - \lambda_1 I) = \text{Span}\{\vec{v}_1, \vec{v}_2\} \quad (1)$$

are bases

$$N(A - \lambda_2 I) = \text{Span}\{\vec{w}_1, \vec{w}_2\} \quad (2)$$

MUST SHOW $\{\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2\}$ is LI

Suppose

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \beta_1 \vec{w}_1 + \beta_2 \vec{w}_2 = \vec{0}. \quad (3)$$

NOW

$$\vec{0} = (A - \lambda_2 I)(-\beta_1 \vec{w}_1 + -\beta_2 \vec{w}_2) \quad \text{by } (2)$$

$$= (A - \lambda_2 I)(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) \quad \text{by } (3)$$

$$= [(A - \lambda_1 I) + (\lambda_2 - \lambda_1)](\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2)$$

$$= (\lambda_2 - \lambda_1)(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) \quad \text{by } (1)$$

Since $\lambda_2 \neq \lambda_1$ get

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = \vec{0}$$

So $\alpha_1 = \alpha_2 = 0$ by (1)

So $\beta_1 \vec{w}_1 + \beta_2 \vec{w}_2 = 0$ by (3)

So $\beta_1 = \beta_2$ by (2) $\square$

3 If A has no repeated evalues

Then A has n distinct evalues

So by 1 $\exists_n$ Evectors That form L.I. set

So A has complete set of evectors

So A is diagonalizable

4 IDEA BY EXAMPLE FOR $\boxed{\Leftarrow}$

Suppose A is 5×5

and $p(\lambda) = (\lambda - \lambda_1)^3 (\lambda - \lambda_2)^2$ $\sigma(A) = \{\lambda_1, \lambda_2\}$

and $N(A - \lambda_1 I) = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$
$N(A - \lambda_2 I) = \text{Span}\{\vec{w}_1, \vec{w}_2\}$ } (BASIS)

In This EX $\text{Alg Mult}_A(\lambda_1) = 3 = \text{Geo Mult}_A(\lambda_1)$

$\text{Alg Mult}_A(\lambda_2) = 2 = \text{Geo Mult}_A(\lambda_2)$

Then by 2 $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}_1, \vec{w}_2\}$ is complete Set of Evectors

∎

E APPLICATION TO ODE's [SCAFFOLDED PROBLEM]

Let $A \in \mathbb{C}^{n \times n}$ be DIAGONALIZABLE, $A = PDP^{-1}$

Consider IVP

$$\begin{cases} \frac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

(a) Let $\vec{u} = P^{-1}\vec{x}$
What IVP does $\vec{u}$ satisfy?

(b) Solve the IVP for $\vec{u}$ by using fact it decomposes into n separate IVP's for the components $u_1, \dots, u_n$ of $\vec{u}$.

(c) Define $e^{Dt} = \text{Diag}\left[e^{\lambda_1 t}, \dots, e^{\lambda_n t}\right]$

Show from (b) that $\vec{u}(t) = e^{Dt}\vec{u}_0$.

(d) Hence obtain formula for $\vec{x}(t)$ using (a)

(e) Using $A = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}$, $\vec{x}_0 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ see that you get same answer as in 7.1 (c).