7.1 EIGENSYSTEMS [BRIEF VERSION]

(A) MOTIVATION FROM ODE's

(i) SCALAR ODE's

RECALL: IF $x : [0, \infty) \to \mathbb{R}$ SATISFIES IVP

$$\begin{cases} \dfrac{dx}{dt} = ax \\ x(0) = x_0 \end{cases} \qquad (1)$$

Then $x(t) = x_0 e^{at}$

(ii) VECTOR-VALUED ODE's

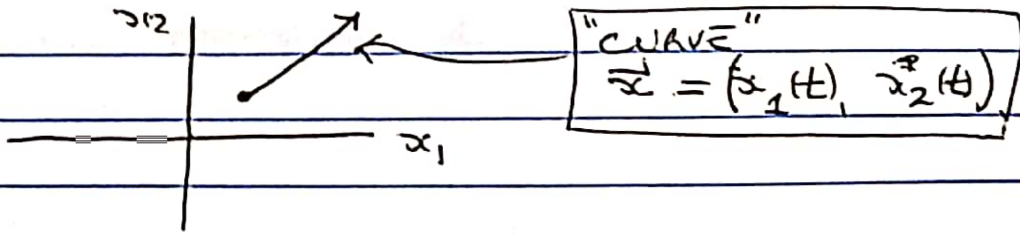
SUPPOSE ~~IF~~ $\vec{x} : [0, \infty) \to \mathbb{R}^n$ satisfies IVP

$$\begin{cases} \dfrac{d\vec{x}}{dt} = A\vec{x} \quad \text{where } A \text{ is } n \times n \\ \vec{x}(0) = \vec{x}_0 \end{cases} \qquad (2)$$

EX

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix} \vec{x} \iff \begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= 6x_1 - x_2 \end{aligned} \qquad (3)$$

OFTEN $\exists$ STRAIGHT LINE SOLUTIONS:



$\boxed{\text{MOTIVATION + INTERPRETATION}}$ 1B

Consider $x = x(t)$ satisfying

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 6x = 0 \quad (1)$$

with initial conditions

$$x(0) = x_0$$

$$\frac{dx}{dt}(0) = v_0.$$

SET $y = \frac{dx}{dt}$

Then (1) says

$$\frac{dy}{dt} + y - 6x = 0$$

$$\frac{dx}{dt} - y = 0$$

OR

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad (2)$$

$$\begin{pmatrix} x \\ y \end{pmatrix}(0) = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

$$\boxed{\frac{d\vec{x}}{dt} = A\vec{x}}$$

CLAIM $\vec{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $c_1, c_2 \in \mathbb{R}$ (3)

is (general) solution to (2)

1c

CHECK

$$\frac{d\vec{x}}{dt} = c_1 e^{2t}\, 2\begin{pmatrix}1\\2\end{pmatrix} \mp c_2 e^{-3t}\, 3\begin{pmatrix}-1\\3\end{pmatrix}$$

$$A\vec{x} = c_1 e^{2t} A\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-3t} A\begin{pmatrix}-1\\3\end{pmatrix}$$

and

$$A\begin{pmatrix}1\\2\end{pmatrix} = 2\begin{pmatrix}1\\2\end{pmatrix}, \qquad A\begin{pmatrix}-1\\3\end{pmatrix} = 3\begin{pmatrix}-1\\3\end{pmatrix}$$

is easy to check

$\square$

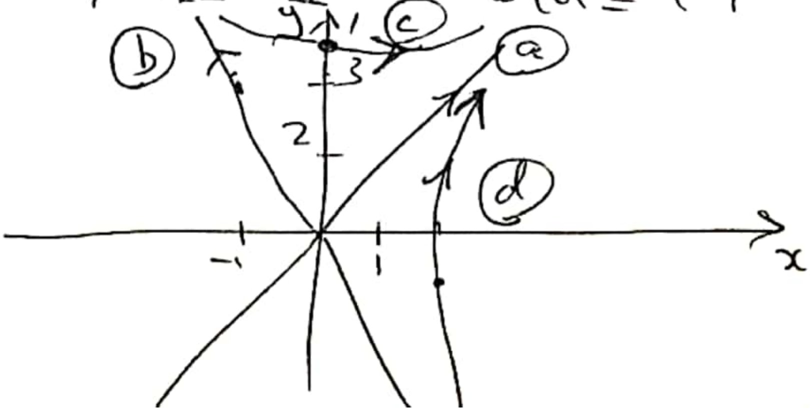
NOTICE By 3

(a) $\lim_{t \to +\infty} \vec{x}(t) = c_1 e^{2t}\begin{pmatrix}1\\2\end{pmatrix}$ ← STRAIGHT LINE SOLNS

(b) $\lim_{t \to -\infty} \vec{x}(t) = c_2 e^{-3t}\begin{pmatrix}-1\\3\end{pmatrix}$

(c) IF $c_1 = c_2 = 1$ then $\vec{x}(0) = \begin{pmatrix}0\\5\end{pmatrix}$

(d) IF $c_1 = 1,\ c_2 = -1$ $\vec{x}(0) = \begin{pmatrix}2\\-1\end{pmatrix}$



GUESS FORMULA

$$\vec{x}(t) = f(t)\,\vec{v} \quad (4) \qquad f : [0,\infty) \to \mathbb{R}, \quad \vec{v} \in \mathbb{R}^n \text{ FIXED.}$$

PLUG INTO (3)

$$f'(t)\,\vec{v} = A\, f(t)\,\vec{v}$$

$$\Rightarrow \quad \underbrace{\frac{f'(t)}{f(t)}}_{\text{SCALAR}}\, \vec{v} = A\,\vec{v}$$

t – INDEPENDENT

FORCES

$$\frac{f'(t)}{f(t)} = \lambda = t\text{ – INDEPT CONSTANT}$$

SO GET

$$f'(t) = \lambda f(t) \quad (5)$$

AND

$$A\vec{v} = \lambda \vec{v} \quad (6)$$

LIKE SEPARATION OF VARIABLES IN PDEs

SOLUTION OF (5) IS $f(t) = f_0 e^{\lambda t}$

(6) IS AN EIGENVALUE – EIGENVECTOR PROBLEM FOR $n \times n$ MATRIX A

B THE EIGENPROBLEM

DEF 1 GIVEN $n \times n$ A IF $\lambda \in \mathbb{C}$ AND $0 \neq \vec{v} \in \mathbb{C}^n$ SATISFY

$$A\vec{v} = \lambda \vec{v}$$

WE CALL
- λ AN EIGENVALUE OF A
- $\vec{v}$ AN EIGENVECTOR OF A
- $(\lambda, \vec{v})$ IS AN EIGENPAIR
- $\sigma(A)$ = SET OF ALL EIGENVALUES OF A
  = SPECTRUM OF A.

OBSERVE

$$A\vec{v} = \lambda\vec{v} \iff (A - \lambda I)\vec{v} = \vec{0}$$
$$\iff \vec{v} \in N(A - \lambda I)$$

$\therefore$

THM 2

1. $\lambda \in \sigma(A) \iff N(A - \lambda I) \neq \{\vec{0}\}$
   $\iff A - \lambda I$ IS NOT INVERTIBLE
   $\iff p(\lambda) := \det(A - \lambda I) = 0$
   – CHARACTERISTIC EQN FOR A

2. THE SET OF EIGENVECTORS OF A WITH EVALUE λ
   = NON-ZERO ELTS OF $N(A - \lambda I)$

   λ – EIGENSPACE OF A

(c) SCAFFOLDED PROBLEM TO SOLVE ODE SYSTEM

$$\begin{cases} \dfrac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix} \vec{x} \\ \vec{x}(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \end{cases} \qquad A = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}$$

(a) SHOW THAT SOLUTIONS OF FORM $\vec{x}(t) = e^{\lambda t}\vec{v}$ CORRESPOND TO EIGENPAIRS $(\lambda, \vec{v})$ OF A

(b) SOLVE CHARACTERISTIC EQN $0 = p(\lambda) = \det(A - \lambda I)$ TO DETERMINE EIGENVALUES.

HINT: A is 2×2 SO $p(\lambda)$ = DEG 2 POLY.

(c) FOR EACH λ CALCULATE $N(A - \lambda I)$ USING G.E.

YOU WILL GET

$$\lambda = 2 \qquad N(A - 2I) = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$$
$$\lambda = -3 \qquad N(A + 3I) = \text{Span}\left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right\}$$

SO $\exists$ BASIS OF $\mathbb{R}^2$ CONSISTING OF EVECTORS OF A

(d) LATER we will show every solution of ODE is of form

$$\vec{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} \qquad c_j \in \mathbb{C}.$$

Find particular solution solving $\vec{x}(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

HINT: SET UP + SOLVE 2×2 LINEAR SYSTEM FOR $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

(e) SKETCH $\vec{x}(t)$ IN (x_1, x_2) — SPACE.

D THEORY ABOUT EIGENVALUES

THM 3
Let A be $n \times n$

1. $p(\lambda) = \det(A - \lambda I)$ is a deg n POLY with leading term $(-1)^n \lambda^n$.

So by FUNDAMENTAL THM OF ALGEBRA

2. A has n eigenvalues
   - Complex and/or Repeated possible.
3. IF $A \in \mathbb{R}^{n \times n}$ Then evalues come in complex conjugate pairs.

PROBLEM: GALOIS THEORY (Abstract Algebra II) tells us $\nexists$ formula for λ's. for $n \geq 5$

SOLUTION: Use numerical linear algebra algorithms to approximate λ's. (MATLAB)

WITH LUCK CAN USE $(n = 3 \text{ or } 4)$

THM 4

IF α_i's are INTEGERS THEN every INTEGER solution of

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0 = 0$$

must be a factor of α_0.

(E) SCAFFOLDED PROBLEMS

(1) FIND ALL ROOTS OF $p(\lambda) = \lambda^3 + 3\lambda^2 + 5\lambda + 3$.

(a) WHAT IS α_0?

(b) WHAT DOES THM 4 TELL US?

(c) USE LONG DIVISION OF POLYNOMIALS TO FACTOR $p(\lambda)$ AS PRODUCT OF A LINEAR FUNCTION AND A QUADRATIC.

(d) FINISH OFF THE PROBLEM.

DEF A <u>COMPLETE SET OF EIGENVECTORS</u> FOR $A \in \mathbb{C}^{n \times n}$ IS A SET OF n EIGENVECTORS OF A THAT FORM A BASIS FOR $\mathbb{C}^n$.

NOTE There is no guarantee a matrix A has a complete set of eigenvectors.

(2) LET $A = \begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$

(a) SHOW $\sigma(A) = \{3, -2\}$.
Which eigenvalue is repeated?

(b) Calculate $N(A - 3I)$ and $N(A + 2I)$

(c) Can you find a complete set of eigenvectors for A?

3 COMPLEX EPAIRS + ODE's

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x} \qquad (*)$$

(a) By calculating epairs of $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ show general solution is

$$\vec{x}(t) = c_1 e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad c_j \in \mathbb{C}.$$

(b) The ODE is real but solution is complex. YUK!!

Use Euler's Formula $e^{it} = \cos t + i \sin t$ to show

$$\vec{x}(t) = a_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + a_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \qquad a_j \in \mathbb{C}.$$

(c) SOLVE IVP with $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(d) SKETCH SOLUTION IN (c) IN (x_1, x_2)-PLANE.

(e) Show how solution to $(*)$ is related to 2nd order ODE

$$u'' + u = 0 \qquad u = u(t)$$
$$u : \mathbb{R} \to \mathbb{R}.$$

F $\boxed{7.1.18}$ A is $n \times n$

a $\boxed{\sigma(A^T) = \sigma(A)}$

This means A and A^T have same evalues with same algebraic multiplicities.

PF

$$\det(A^T - \lambda I) = \det((A - \lambda I)^T)$$

$$= \det(A - \lambda I) \quad \text{as } \det B^T = \det B$$

b $\boxed{\lambda \in \sigma(A) \Leftrightarrow \bar{\lambda} \in \sigma(A^*)}$ $\quad A^* = \bar{A}^T$

EG if $\lambda = 5 + i$ is evalue of A with alg mult 4

THEN $\bar{\lambda} = 5 - i$ is evalue of A^* with alg mult 4

PF $\lambda \in \sigma(A) \Leftrightarrow \det(A - \lambda I) = 0$

$$\Leftrightarrow \det((A - \lambda I)^T) = 0$$

$$\Leftrightarrow \overline{\det((A - \lambda I)^T)} = 0$$

$$\Leftrightarrow \det\left(\overline{(A - \lambda I)^T}\right) = 0$$

as $\det(\bar{B}) = \overline{\det(B)}$

$$\Leftrightarrow \det((A - \lambda I)^*) = 0$$

$$\Leftrightarrow \det(A^* - \bar{\lambda} I) = 0 \Leftrightarrow \bar{\lambda} \in \sigma(A^*)$$

(c) IF $A \in \mathbb{R}^{n \times n}$

Then $\lambda \in \sigma(A) \iff \bar{\lambda} \in \sigma(A)$

ie Evalues come in complex conjugate pairs

EX If A is real 3×3 and $\lambda = 5+2i$ is evalue

Then $\lambda = 5-2i$ must also be evalue

Q If A is $n \times n$ where n is ODD what can you conclude about how many evalues of A must be real?

Pf of (c)

By (b)

$$\lambda \in \sigma(A) \iff \bar{\lambda} \in \sigma(A^*)$$

$$\begin{aligned} \sigma(A^*) &= \sigma(A^T) && \text{as } A \text{ is Real} \\ &= \sigma(A) && \text{by (a)} \end{aligned}$$

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