6.1 DETERMINANTS

1

Recall For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det(A) = ad - bc.$$

POEM

Determinants used to be hot
Now they are not
~~But they are still useful~~

They have lovely properties
But hideous formulas

We will learn how to define + compute det of $\underline{\underline{n \times n}}$ matrix A

MAIN USES

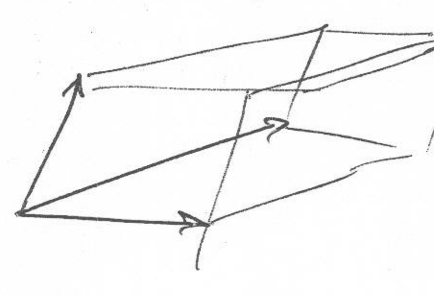
① $\det : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$ Characterizes A using a single number.

$$\det(A) \neq 0 \iff A \text{ is Invertible.}$$

② dets are used to ~~define~~ characterize eigenvalues. (#7)

③ Let P be parallelopiped in $\mathbb{R}^n$ whose n edge vectors are the ~~rows~~ cols of A.

Then $\det(A) = n$-dimensional volume of P



(4) $\det(A) = \pm$ Product of Pivots

(5) Determinants are used to measure how much solution $\vec{x}$ of $A\vec{x} = \vec{b}$ changes when we change one of the elements of $\vec{b}$.

I don't like Meyer's defⁿ of det.

Instead I will follow Strang "Linear Algebra + Its Apps".

I WANT YOU TO USE THIS APPROACH TOO.

<u>DEFN</u> The determinant ~~of an $n \times n$ matrix A~~ is a function from the vector space of $n \times n$ real matrices to $\mathbb{R}$ ~~is the scalar~~ that is characterized by the following 3 properties

(I) The determinant depends linearly on the first row.

(II) The determinant changes sign when two rows are exchanged

(III) The determinant of the identity matrix is 1.

IE USING these 3 properties we can prove $\exists!$ # $\det(A)$

Explanation of I

If $A = \begin{bmatrix} \vec{v} \\ \hline \vec{u}_2 \\ | \\ \vec{u}_n \end{bmatrix}$ $B = \begin{bmatrix} \vec{w} \\ \hline \vec{u}_2 \\ | \\ \vec{u}_n \end{bmatrix}$ are identical from ~~first~~ 2nd row down

and

$$C = \begin{bmatrix} \alpha \vec{v} + \beta \vec{w} \\ \hline \vec{u}_2 \\ | \\ \vec{u}_n \end{bmatrix}$$

Then $\boxed{\det(C) = \alpha \det(A) + \beta \det(B)}$

LET'S CHECK I, II, III for 2x2 matrices using old defn

I $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $B = \begin{pmatrix} e & f \\ c & d \end{pmatrix}$

$$C = \begin{pmatrix} \alpha a + e & \alpha b + f \\ c & d \end{pmatrix}$$

$$\begin{aligned} \det(C) &= (\alpha a + e)d - (\alpha b + f)c \\ &= \alpha(ad - bc) + ed - fc \\ &= \alpha \det(A) + \det(B) \checkmark \end{aligned}$$

II $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

$\det(B) = \det\begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - ad = -(ad - bc) = -\det(A)$ ✓

III $\quad \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.1 - 0.0 = 1$ ✓

— o —

$\boxed{\text{CAUTION}}$ Even though I is true for arbitrary matrices A, B

$$\det(A+B) \neq \det(A) + \det(B)$$

EX

$A = I, \quad B = -I$

$A + B = 0$

$\det(0) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$\det(I) = 1$

$\det(-I) = \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}$

COUNTEREXAMPLE

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\det A = 1 \qquad \det B = 4 - 6 = -2$

$\det(A+B) = \det\begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix} = 10 - 6$
$= 4.$

To use these 3 rules to find det (A) we need to derive some more properties from I — III.

IV If two rows of A are equal then det A = 0.

PF

Suppose Rows k and l of A are equal.

Obtain B by switching rows k and l of A.

Clearly $B = A$! ~~So det (B) =~~

But by II $\det(B) = -\det(A)$ holds.

So $\det(A) = -\det(A) \Rightarrow \det(A) = 0$

V The determinant depends linearly on the kth row

PF You do it using II, II and II again

VI* Suppose we get B from A doing the elementary row opn

$$\boxed{\text{Row } k = \text{Row } k - \alpha \, \text{Row}(l)}$$

Then $\det(B) = \det(A)$

PF $n=2$ Case gives idea $\qquad R2 = R2 - \alpha R1$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} a-\alpha c & b-\alpha d \\ c & d \end{pmatrix}$$

$$\det B \overset{\mathrm{I}}{=} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \alpha \det \begin{pmatrix} c & d \\ c & d \end{pmatrix}$$

$$\overset{\mathrm{IV}}{=} \det(A) - \alpha \cdot 0 = \det(A) \qquad \blacksquare$$

VII If A has a zero row, then $\det(A) = 0$.

PF $n=2$ gives idea again

$$\det \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \underset{\mathrm{VI}}{\overset{R1=R1+R2}{=}} \det \begin{pmatrix} c & d \\ c & d \end{pmatrix} \overset{\mathrm{IV}}{=} 0.$$

~~VIII~~ DEF A matrix A is upper triangular if all elts below diagonal are zero:

$$A = \begin{bmatrix} * & \cdots & * \\ 0 & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots \; 0 & * \end{bmatrix} \quad \text{or} \quad A_{ij} = 0 \text{ for } i > j$$

$\uparrow$ A_{n1}

VIII If A is upper (or lower) triangular, then

$$\det(A) = A_{11} A_{22} \cdots A_{nn} = \text{Product of diag entries}$$

PF ~~Repe~~ CASE I Diagonal entries of A are all non zero.

Then if we repeatedly use the row opn Row k = Row k $-\alpha$ Row l

we can reduce A to ~~the~~ diagonal ~~matrix~~ $D = \begin{pmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{nn} \end{pmatrix}$

By VI $\det(A) = \det(D)$

By I, V $\det D = A_{11} \det \begin{pmatrix} 1 & & & \\ & A_{22} & & 0 \\ & & \ddots & \\ 0 & & & A_{nn} \end{pmatrix}$

$= \cdots = A_{11} \text{---} A_{nn} \det(I)$

$= A_{11} \text{---} A_{nn}$ by III.

CASE II At least one of diag entries of A is zero.

~~Rk~~ $R_k = R_k - \alpha R_l$ then yields a zero row.

So $\det(A) = 0$ by VI and IV $\square$

IX Let U be the row echelon form of A obtained ~~from~~ using row ops

- Swap two rows
- Row k = Row k $- \alpha$ Row l

COR 2 ** $\det A$ is chard by I — III

Then ① If A is SINGULAR, $\det(A) = 0$ as Elim gives a zero row.

② If A inv, $\det(A) = (-1)^{\sigma} \det(U) \overset{\text{VIII}}{=} (-1)^{\sigma} d_1 \text{---} d_n$

where
- σ = # Row Interchanges
- $d_1 \text{---} d_n$ are PIVOTS. (Non zero)

COR 1 *** $\det(A) = 0$ iff A

PF VI + VIII

EX

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \\ 2 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 6 & 8 \\ 0 & -3 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3-8/3 \\ 0 & 6 & 8 \\ 0 & 0 & -2 \end{pmatrix}$$

So $\det(A) = 1.6.(-2) = -12.$

QUICKEST WAY TO CALC DETS!

Cofactors

$$\det(A) = 1\begin{vmatrix} 4 & 5 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} -1 & 5 \\ 2 & 0 \end{vmatrix} + 3\begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix}$$

$$= -5 + 20 + 3(-1-8) = -12 \checkmark$$

— o —

(X) IF A, B $n \times n$ THEN

$$\boxed{\det(AB) = \det(A)\det(B)}$$

(XI) $\det(P^{-1}CP) = \det(C).$

So just like for trace, we can define det of a Linear Operator $T: V \longrightarrow V$ from a VS to itself.

(XII) $\det(A^T) = \det(A)$

PF OMIT, tho not too hard

(XIII) $\det(A^{-1}) = \dfrac{1}{\det A}$ PF Apply (X) and (III) to $AA^{-1} = I$

Pf of ⊗

1. If ~~A or~~ B ~~are~~ is singular then AB is singular by $N(B) \subseteq N(AB)$ (See Exam)

So $\det(AB) = \det A \det B$ is $0 = 0$

2. ELEGANT PROOF when ~~A,~~ B nonsingular.

Let $$d(A) = \frac{\det(AB)}{\det(B)}$$

Let show d has props I – III

Hence since these props characterize (determine) det A we must have

$$d(A) = \det(A)$$

$$\therefore \det(A) = \frac{\det(AB)}{\det(B)}$$

Checks

III $A = I$: $d(I) = \dfrac{\det(IB)}{\det B} = 1$ ✓

II Recall $(AB)_{i*} = A_{i*}B$

Row i of AB = (Row i of A) B

Hence switching two rows of AB is same as ~~is same~~ switching those two rows of A. So I is true for $d(A)$ as true for $d(AB)$

I Similar argument to II

∎

~~XIV $\det(A^T) = \det(A)$~~

~~PF~~ OMIT.

This means all props ~~I – III~~ hold for cols as well as for rows.

NOTICE 13 PROPERTIES but no EXPLICIT FORMULA YET!

Q Why might we even want a formula?

A So we can find out how much $\det(A)$ changes when we change one entry of A.

EX YOU KNOW

$\boxed{n=2}$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \overset{(12)}{a_{11}a_{22}} - a_{12}a_{21}$$

$\boxed{n=3}$

$$\det\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= \overset{(123)}{a_{11}a_{22}a_{33}} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31}$$

$$\overset{(132)}{-\,a_{11}a_{23}a_{32}} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Lets derive a formula for det(A) using I – III :

CASE n=2 det A = |A|.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{\text{I}}{=} \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$\overset{\text{V}}{=} \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$2 \times 2 = 2^2$ terms

$$\overset{\text{VII}}{\underset{\text{V}}{=}} 0 + ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 0$$

$$\overset{\text{II}}{=} ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - bc \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \overset{\text{III}}{=} ad - bc.$$

COOL

So for $n \times n$ matrix the idea is

1. Write each row as a sum of n vectors in the n coord dirns.
2. Use linearity of det in each row to write det (A) as sum of n^n dets.
3. Most of these dets are 0.
   Which are non-zero?
   For each det
   We ~~get~~ have ONE NON ZERO entry in each row.
   To get a non-zero det we also EXACTLY need 1 entry in each col (otherwise we have a col of 0's)
   ~~So we~~ need a

How many ways can we do this?

1st Row : The entry can go in any of n cols.

2nd Row : $n-1$ cols

|

nth row : 1 col

Total # Ways $= n(n-1) \cdots 2.1 = n!$

NOTE

These $n!$ ways are the $n!$ permutations of the numbers $1,2,\ldots,n$.

$\boxed{n=3}$ $3! = 6$ terms.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix}$$

$$+ \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

$$\det(A) \overset{*}{=} a_{11}a_{22}a_{33}\begin{vmatrix}1&&\\&1&\\&&1\end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix}&1&\\&&1\\1&&\end{vmatrix}$$

$$+ a_{13}a_{21}a_{32}\begin{vmatrix}&&1\\1&&\\&1&\end{vmatrix} + a_{11}a_{23}a_{32}\begin{vmatrix}1&&\\&&1\\&1&\end{vmatrix}$$

$$+ a_{12}a_{21}a_{33}\begin{vmatrix}&1&\\1&&\\&&1\end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix}&&1\\&1&\\1&&\end{vmatrix}$$

To write down an explicit formula in general we need:

DEF A <u>Permutation</u> of $(1, 2, \ldots, n)$ is a rearrangement $p = (p_1\ p_2 \cdots p_n)$ of these #'s.
There are $n!$ perms of $1 - n$.

EG

$(1\ 2\ 3)$ $\qquad$ $(1\ 3\ 2)$
$(2\ 3\ 1)$ $\qquad$ $(2\ 1\ 3)$
$(3\ 1\ 2)$ $\qquad$ $(3\ 2\ 1)$ $\qquad$ in same order as $*$

Let P_p = Permutation Matrix with ~~each~~ kth row having single 1 in col $p(k)$.

Rev II

CLAIM Let $\sigma(p)$ = # Row exchanges req'd to

FORMULA FOR det(A)

$$\det(A) = \sum_{p} A_{1p_1} A_{2p_2} A_{3p_3} \cdots A_{np_n} \det(P_p)$$

$$= \sum_{p} \sigma(p)\, A_{1p_1} A_{2p_2} \cdots A_{np_n}$$

– Meyer uses this formula as def^n.

Ex $\boxed{n=3}$

$\cancel{\sigma}(1\ 3\ 2) \longrightarrow (1\ 2\ 3) \qquad s = 1 \qquad \sigma = -1$

$(3\ 1\ 2) \longrightarrow (1\ 3\ 2) \longrightarrow (1\ 2\ 3) \qquad s = 2 \quad \sigma = +1$

$(3\ 2\ 1) \longrightarrow (1\ 2\ 3) \qquad s = 1. \qquad \sigma = -1$

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0 \quad \text{\underline{as}}$$

p	$\sigma(p)$	$A_{1p_1}\ A_{2p_2}\ A_{3p_3}$	
(1 2 3)	+	$1 \cdot 5 \cdot 9 = 45$	
(1 3 2)	−	$1 \cdot 6\ 8 = 48$	2nd Row pick 3rd col
(2 1 3)	$\cancel{+}$ −	$2\ 4\ 9 = 72$	
(2 3 1)	+	$2\ 6\ 7 = 84$	
(3 1 2)	+	3 [illegible]	

DETERMINANTS OF BLOCK MATRICES

THM 1 Let A, D be square matrices

$$\det\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(A)\det(D)$$

~~PF IDEA~~

EX [on GT]

$$\det\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \\ \hline 0 & 0 & 7 \end{array}\right) \overset{R2=R2-3R1}{=} \det\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -2 & * \\ \hline 0 & 0 & 7 \end{array}\right)$$

$$= (1 \cdot -2) \cdot 7 = -14$$

$$= \det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \det(D)$$

PF IDEA

In genl do row opns to make A upper triangular. This does not change D at all.

Then do row opns to make D upper triangular. The A-block is not changed by this.

So we get an upper triangular matrix with

$$\det\begin{pmatrix} \alpha_{11} & & & * \\ & \alpha_{nn} & & \\ & & \beta_{11} & \\ 0 & & & \beta_{mm} \end{pmatrix} = (\alpha_{11} \cdots \alpha_{nn})(\beta_{11} \cdots \beta_{mm}) = \det(A)\det(D)$$

THM 2 If A, D are square matrices Then

$A_{n\times n}$ $D_{m\times m}$
$C_{m\times n}$ $B_{n\times m}$
$CA^{-1}B$ is
$(m\times n)(n\times n)(n\times m)$
$= m\times m$

① If A^{-1} $\exists$

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B)$$

② If D^{-1} $\exists$

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\det(A - BD^{-1}C)$$

PF

① TRICK

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

[(2,2) BLOCK is $CA^{-1}B + D - CA^{-1}B = D$.]

Then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \underset{\text{RULE}}{\overset{\text{PROD}}{=}} \det\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \det\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

$$\underset{\substack{+\det E = \\ \det E^T}}{\overset{\text{THM 1}}{=}} 1 \cdot \det(A)\det(D - CA^{-1}B)$$

~~6.4 MORE ON DET~~

RANK – ONE UPDATES

Let $\vec{c}, \vec{d}$ be $n \times 1$ column vectors.

Then $B = \vec{c}\,\vec{d}^T$ is $n \times n$ $\quad B_{ij} = c_i d_j$

$$\mathrm{Range}(B) = \{ \vec{c} \underbrace{\vec{d}^T \vec{x}}_{1\times 1} \mid \vec{x} \in \mathbb{R}^n \} \qquad \vec{x} \text{ is } n \times 1$$

$$= \{ (\vec{d}^T \vec{x})\, \vec{c} \mid \vec{x} \in \mathbb{R}^n \}$$

$$= \mathrm{Span}(\vec{c}) \quad \text{provided } \vec{d} \neq \vec{0}.$$

So $\boxed{\mathrm{Rank}(\vec{c}\,\vec{d}^T) = 1}$ if $\vec{c}, \vec{d}$ are nonzero

NOTICE $\mathrm{Trace}(B) = \sum_{i=1}^{n} c_i d_i = \underbrace{\vec{d}^T \vec{c}}_{1\times 1}$

THM

1. $\det(I + cd^T) = 1 + \vec{d}^T \vec{c} = 1 + \mathrm{Trace}(\vec{c}\,\vec{d}^T)$

2. If A is $n \times n$ invertible then

$$\boxed{\det(A + \vec{c}\,\vec{d}^T) = \det(A)\left(1 + \vec{d}^T A^{-1} \vec{c}\right)}$$

PF

(1) Notice $1+d^Tc$ is 1×1 matrix.

So we ~~have~~ ~~want~~ have to prove

$$\det(\underbrace{I+cd^T}_{n\times n}) = \det(\underbrace{1+d^Tc}_{1\times 1})$$

One way to prove 2 matrices have same det is to show they are similar. But for that they need to have same size.

TRICK Build a bigger ~~matrix~~.

CLAIM

$$B = \begin{array}{c|c} & \begin{array}{cc} n & \quad 1 \end{array} \\ \begin{array}{c} n \\ 1 \end{array} & \left(\begin{array}{c|c} I+cd^T & c \\ \hline 0 & 1 \end{array}\right) \end{array} \text{ is similar to } C \begin{array}{c|c} & \begin{array}{cc} n & \quad 1 \end{array} \\ \begin{array}{c} n \\ 1 \end{array} & \left(\begin{array}{c|c} I & c \\ \hline 0 & 1+d^Tc \end{array}\right) \end{array}$$

PROOF

~~First check that~~

To do this we'll need to multiply d^T by c (*).

So consider

$$P = \begin{pmatrix} I & 0 \\ d^T & 1 \end{pmatrix} \quad \text{U☐✓} \quad P^{-1} = \begin{pmatrix} I & 0 \\ -d^T & 1 \end{pmatrix}$$ (Could use GJE)

~~Check~~

U☐✓ $\qquad PBP^{-1} = C$

(5)

(2) $A + cd^T = A(I + \underbrace{A^{-1}c}_{n\times 1} d^T)$

So $\det(A + cd^T) \overset{PR}{=} \det(A)\det(I + \underbrace{A^{-1}c}\, d^T)$

$\overset{(1)}{=} \det(A)(1 + d^T A^{-1} c)$

CRAMER'S RULE

Let A be $n\times n$, Invertible, $\vec{x} = \begin{pmatrix} x_1 \\ | \\ x_n \end{pmatrix}$.

The sol^n of $A\vec{x} = \vec{b}$ is given by

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1 — n$$

where A_i is identical to A except i-th col has been replaced by $\vec{b}$:

$$A_i = [A_{*1} \mid A_{*2} \ldots \mid A_{*\,i-1} \mid \vec{b} \mid A_{*\,i+1} — A_{*n}]$$

$\boxed{\text{SOL}^{\text{N}}\ \vec{x}\ \text{IS RATIO OF 2 POLYS, } \cancel{\text{WHOSE ENTRIE}} \text{ IN ENTRIES of } A, \vec{b}.}$

~~EXAMPLE~~ Get A_i from A by making rank 1 update (change 1 col) ~~of~~ A.

PROOF $A_i = A + \underbrace{(b - A_{*i})\, e_i^T}_{\text{RANK 1 UPDATE TO } A}$ as $A = \sum_{i=1}^{n} \underbrace{A_{*i}}_{\text{COL } i} e_i^T$

So

$\det(A_i) = \det(A)[\,1 + e_i^T A^{-1}(b - A_{*i})\,]$ $\qquad A^{-1}A = I$

$= \det(A)[\,1 + e_i^T(\vec{x} - e_i)\,]$ $\qquad$ ith col is

$= \det(A)\,[1 + x_i - 1] = x_i \det(A)$ ⑥

When is Cramer's Rule Useful?

- When only need 1 component of $\vec{x}$.
- When $A, \vec{b}$ depend on a parameter t and we want to understand how $\vec{x}$ depends on t.
  - We would like analytic formula for $\vec{x}(t)$. Cramer's Rule gives it to us!

OMIT

EX In $\mathbb{R}^2$

START HERE

COFACTORS

We know det(A) depends linearly on 1st row.

SINCE

$$[a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] = \sum_{i=1}^{n} a_{1i} \vec{e}_i^{\,T}$$

$$\det A = a_{11} \det\begin{pmatrix} \vec{e}_1^{\,T} \\ \hline A_{2*} \\ | \\ A_{n*} \end{pmatrix} + a_{12} \det\begin{pmatrix} \vec{e}_2^{\,T} \\ \hline A_{2*} \\ | \\ A_{n*} \end{pmatrix} + \cdots + a_{1n} \det\begin{pmatrix} \vec{e}_n^{\,T} \\ \hline A_{2*} \\ | \\ A_{n*} \end{pmatrix}$$

Let M_{ij} = $(n-1)\times(n-1)$ MINOR obtained by deleting row i, NOW

$$\left|\begin{array}{c} \vec{e}_1^{\,T} \\ \hline A_{2*} \\ | \\ A_{n*} \end{array}\right| = \left|\begin{array}{c|c} 1 & 0 \\ \hline * & \\ | & M_{11} \\ * & \end{array}\right|$$

$$\overset{\text{BLOCK}}{\underset{\text{DET}}{=}} \; 1 \cdot \det(M_{11})$$

AND

$$\left|\begin{array}{c} \vec{e}_2^{\,T} \\ \hline A_{2*} \\ | \\ A_{n*} \end{array}\right| = \left|\begin{array}{c|c|c} 0 & 1 & 0 \text{ --- } 0 \\ \hline & * & \\ (M_{12})_{*1} & | & (M_{12})_{*(2-n)} \\ & * & \end{array}\right|$$

$$\overset{\text{SWAP}}{\underset{\text{2 COLS}}{=}} - \left|\begin{array}{c|c} 1 & 0 \\ \hline * & \\ | & M_{12} \\ * & \end{array}\right| = -\det(M_{12})$$

DEFN (i,j) – COFACTOR of A is

$$\boxed{\mathring{A}_{ij} = (-1)^{i+j} \det(M_{ij})}$$

FORMULATE

$$\det(A) \overset{\text{ROW } 1}{=} a_{11}\mathring{A}_{11} + a_{12}\mathring{A}_{12} + \cdots + a_{1n}\mathring{A}_{1n}$$

$$\overset{\text{ROW } i}{=} a_{i1}\mathring{A}_{i1} + a_{i2}\mathring{A}_{i2} + \cdots + a_{in}\mathring{A}_{in}$$

$$\overset{\text{col } j}{=} a_{1j}\mathring{A}_{1j} + \cdots + a_{nj}\mathring{A}_{nj}$$

EX Cofactor expansions ~~useful~~ QUICK when have lots of zeros in a row or col

$$|A| = \begin{array}{c} \begin{array}{cccc} - & + & - & + \end{array} \\ \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 0 \\ 7 & 8 & 9 & 10 \\ 0 & 11 & 12 & 0 \end{vmatrix} \end{array} \overset{\text{ROW } 2}{=} -5 \begin{vmatrix} 1 & 2 & 4 \\ 7 & 8 & 10 \\ 0 & 11 & 0 \end{vmatrix}$$

$$= (-5)(-11)\begin{vmatrix} 1 & 4 \\ 7 & 0 \end{vmatrix}$$

$$= 55\,(10 - 28) = 55 \cdot (-18).$$

However when are no zero entries in $A_{n \times n}$, ~~[illegible]~~ ~~requires~~ this method is very slow. ~~when n is larger than~~
Requires over $n!$ mult^{ns}

EX For 10×10 matrix need over 5 million mult^{ns} to calculate $\det(A)$ using cofactors

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For 100×100 need over 10^{150} mult^{ns} ~~[illegible]~~ 1 GHz machine [illegible] 10^{9} mult/sec