5.11 ORTHOGONAL DECOMPOSITION

DEF Let M be a subset of IPS V.
The ORTHOGONAL COMPLEMENT of M in V is

$$M^{\perp} = \{ \vec{x} \in V \mid \langle \vec{m} \mid \vec{x} \rangle = 0 \ \forall \vec{m} \in M \}$$

= Set of vectors $\perp$ to all elts of M.

NOTE M just needs to be a subset not necessarily a subspace of V. But $M^{\perp}$ is always a subspace.

EX

(1) $M = \{\vec{0}\} \Rightarrow M^{\perp} = V$

(2) $M = \text{Span}\{\vec{y}\}$, $V = \mathbb{R}^3$

$M^{\perp}$ = Plane thru $\vec{0}$ with normal $\vec{y}$.

THM 1

(A) If M is a vector subspace of V Then $M^{\perp}$ is a subspace and $V = M \oplus M^{\perp}$

(B) If $N \subset V$ has properties

$$V = M \oplus N$$
$$N \perp M$$

Then $N = M^{\perp}$

(C) $\dim M^{\perp} = \dim V - \dim M$

(D) $(M^{\perp})^{\perp} = M$.

PF

(A) (i) CLAIM $M \cap M^{\perp} = \{\vec{0}\}$

PF If $\vec{x} \in M \cap M^{\perp}$ Then

$\vec{x} \perp \vec{x}$

So $\|\vec{x}\| = 0 \Rightarrow \vec{x} = \vec{0}$.

(ii) $\therefore S = M \oplus M^{\perp}$ is a subspace of V.

Let $B_M, B_{M^\perp}$ be ONB for $M, M^{\perp}$

By Comp Subspace Thm, $B_M \cup B_{M^\perp}$ is ONB for S.

(iii) If $S \neq V$ by Basis Extn Thm and G-S.

$\exists$ ONB $B = B_M \cup B_{M^\perp} \cup \mathcal{E}$ for V.

By constn $\mathcal{E} \perp B_M \Rightarrow \mathcal{E} \perp M$

$\Rightarrow \mathcal{E} \subseteq M^{\perp}$

$\Rightarrow \mathcal{E} \subseteq \operatorname{Span}(B_{M^\perp})$

which contradicts LI of B. So $S = V$ must hold

B OMIT

C Follows from A

D (i) CLAIM $(m^\perp)^\perp \subseteq m$

Pf Let $\vec{x} \in (m^\perp)^\perp \subseteq V$

Since $V = m \oplus m^\perp$

$\vec{x} = \vec{m} + \vec{n}$

NB $\vec{n} = \vec{0}$.

Well $0 = \langle \vec{x} | \vec{n} \rangle$ as $\vec{x} \in (m^\perp)^\perp$ and $\vec{n} \in m^\perp$

$= \langle \vec{m} | \vec{n} \rangle + \langle \vec{n} | \vec{n} \rangle = \|\vec{n}\|^2$.

So $\vec{n} = \vec{0}$. D

(ii) Show $\dim (m^\perp)^\perp = \dim m$ using C x2.

LET $A \in \mathbb{C}^{m \times n}$. So $A^* \in \mathbb{C}^{n \times m}$

Regard

$$\begin{array}{ccccc} & \mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^m & \\ & \cup & \xleftarrow{A^*} & \cup & \\ N(A) & R(A^*) & & N(A^*) & R(A) \end{array}$$

FUNDAMENTAL THM of ADJOINT

Let $A \in \mathbb{C}^{m \times n}$ [ESSENTIALLY NO HYPOTHESES!]

Then

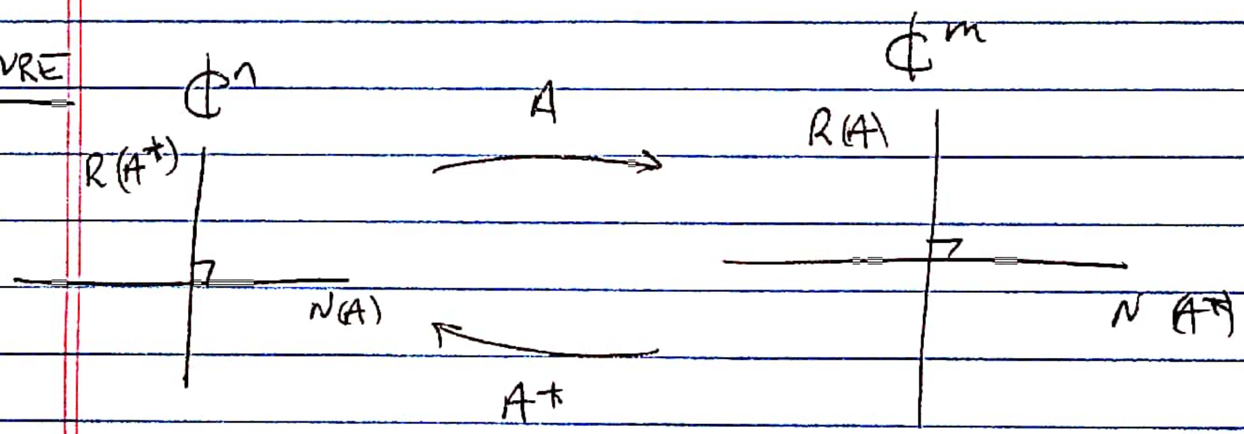
1. $R(A)^{\perp} = N(A^*)$

   So $\mathbb{C}^m = R(A) \oplus N(A^*)$

2. $N(A^{\perp}) = R(A^*)$

   So $\mathbb{C}^n = N(A) \oplus R(A^*)$

PICTURE



PROOF OF 1

$$\vec{x} \in R(A)^{\perp} \iff \langle A\vec{y} \mid \vec{x} \rangle = 0 \quad \forall \vec{y} \in \mathbb{C}^n$$

$$\iff \langle \vec{y} \mid A^* \vec{x} \rangle = 0 \quad \forall \vec{y} \in \mathbb{C}^n$$

$$\iff A^* \vec{x} = \vec{0}$$

$$\iff \vec{x} \in N(A^*)$$

$\square$

Things are even nicer when A is normal.

DEF $A \in \mathbb{C}^{n \times n}$ (Square) is NORMAL if

$$A^* A = A A^*$$

EXS The following matrices are normal

1. Real Symmetric : $A^T = A$
2. Hermitian $A^* = A$
3. Real Anti-symmetric : $A^T = -A$
4. Skew-Hermitian : $A^* = -A$
5. Orthogonal / Unitary : $A^* A = I = A A^*$

The following result is used in pf of Spectral Thm for Normal matrices:

RANGE - NULLSPACE DECOMPOSITION FOR NORMAL MATRICES

Let $A \in \mathbb{C}^{n \times n}$ be normal. Then

1. $R(A)^\perp = N(A)$

2. $\mathbb{C}^n = R(A) \oplus N(A)$

3. $\exists$ UNITARY U and INVERTIBLE C so that

$$\boxed{A = U \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right) U^*}$$

THE PROOF RELIES ON

LEMMA ① $N(A^*A) = N(A)$ and ② $N(AA^*) = N(A^*)$

PF of ①

$\boxed{\supseteq}$ Let $\vec{x} \in N(A)$.

So $A\vec{x} = \vec{0}$

So $A^*A\vec{x} = \vec{0}$

So $\vec{x} \in N(A^*A)$

$\boxed{\subseteq}$ Let $\vec{x} \in N(A^*A)$

So $A^*A\vec{x} = \vec{0}$

So $0 = \langle A^*A\vec{x} \mid \vec{x} \rangle = \langle A\vec{x} \mid A\vec{x} \rangle = \|A\vec{x}\|^2$

So $A\vec{x} = \vec{0}$

So $\vec{x} \in N(A)$

$\square$

PF OF THM

① $R(A)^\perp = N(A^*)$ By Fund Thm Adjoint

$= N(AA^*)$ By Lemma ②

$= N(A^*A)$ as A is normal

$= N(A)$ by Lemma ①

② Follows from ① and Thm 1

3 Let $\{\vec{u}_1 \dots \vec{u}_r\}$ be ONB for $R(A)$

$\{\vec{v}_{r+1}, \dots \vec{v}_n\}$ be ONB for $N(A)$

By 1, 2 and Complementary Subspace Thm

$B = \{\vec{u}_1 \dots \vec{u}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$ is ONB for $\mathbb{C}^n$.

Now

$$A\vec{u}_i = \sum_{j=1}^{r} \langle \vec{u}_j | A\vec{u}_i \rangle \vec{u}_j + \sum_{k=r+1}^{n} \langle \vec{v}_k | A\vec{u}_i \rangle \vec{v}_k$$

$$= C_{ij}\vec{u}_j + \vec{0} \quad \text{as} \quad \langle \underset{N(A)}{\underset{\cap}{\vec{v}_k}} \,|\, \underset{R(A)}{\underset{\cap}{A\vec{u}_i}} \rangle = 0$$

$$N(A) \perp R(A)$$

and

$$A\vec{v}_k = \vec{0} \quad \text{as} \quad \vec{v}_k \in N(A)$$

So

$$[A]_B = \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right)$$

Let $U = [\vec{u}_1 \dots \vec{u}_r, \vec{v}_{r+1} \dots \vec{v}_n]$ is unitary

So

$$A = [A]_E = U[A]_B U^* = U \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right) U^*$$

Finally $r = Rk(A) = Rk \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right)$ where C is $r \times r$

So C is invertible. □