L1 VECTOR SPACES, VECTOR SUBSPACES, LINEAR TRANSFORM^{ns}

[M, 4.1]

DEF 1 A set V is a VECTOR SPACE over $\mathbb{R}$ if $\exists$ operations

$$+ : V \times V \longrightarrow V \qquad \text{ADDITION}$$
$$(\vec{v}, \vec{w}) \longmapsto \vec{v} + \vec{w}$$

$$\cdot : \mathbb{R} \times V \longrightarrow V \qquad \text{SCALAR}$$
$$(\alpha, \vec{v}) \longmapsto \alpha \cdot \vec{v} \qquad \text{MULTIPLICATION}$$

which satisfy 10 AXIOMS which allow one to do algebraic operations on vectors involving combinations of $+$, $\cdot$ just like for real #s, except can't divide by a vector.

EX
$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$
$$\alpha(\vec{u} + \vec{v}) = \alpha \vec{u} + \alpha \vec{v} \qquad \text{etc.}$$

EXS

1. $\mathbb{R}^n$ $\quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad (\alpha \vec{x} + \vec{y})_i := \alpha \vec{x}_i + \vec{y}_i$

2. $\mathbb{R}^{n \times m}$ = Set of $n \times m$ matrices

$$(\alpha A + B)_{ij} := \alpha A_{ij} + B_{ij}$$

3. $C^\infty(\mathbb{R}, \mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \text{ infinitely differentiable} \}$

$$(\alpha f + g)(x) := \alpha f(x) + g(x).$$

Many vector spaces arise as subsets of other vector spaces.

DEF 2 A non-empty subset S of a V.S. V is called a SUBSPACE of V if

$$\forall \alpha \in \mathbb{R},\ \forall \vec{v}, \vec{w} \in S \text{ we have } \alpha \vec{v} + \vec{w} \in S.$$

OPERATIONS DEFINED IN V

PROP^n 3 A subspace of a V.S. is itself a V.S.

DEF 4 The SPAN of a finite set of vectors $\mathcal{S} = \{\vec{v}_1, \dots, \vec{v}_r\} \subseteq V$ is the set

$$S = \operatorname{Span}(\mathcal{S}) = \{\alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r \mid \alpha_1, \dots, \alpha_r \in \mathbb{R}\}$$

PROP 5 For any finite subset $\mathcal{S}$ of V, $\operatorname{Span}(\mathcal{S})$ is a V.S.

PF Let $\alpha \in \mathbb{R}$, $\vec{x}, \vec{y} \in S$. Must show $\alpha \vec{x} + \vec{y} \in S$.
Well

$$\vec{x} = \beta_1 \vec{v}_1 + \dots + \beta_r \vec{v}_r$$
$$\vec{y} = \gamma_1 \vec{v}_1 + \dots + \gamma_r \vec{v}_r$$

So

$$\alpha \vec{x} + \vec{y} = \alpha(\beta_1 \vec{v}_1 + \dots + \beta_r \vec{v}_r) + (\gamma_1 \vec{v}_1 + \dots + \gamma_r \vec{v}_r)$$

$$\overset{\text{Axioms}}{\underset{\text{for } V}{=}} (\alpha \beta_1 + \gamma_1)\vec{v}_1 + \dots + (\alpha \beta_r + \gamma_r)\vec{v}_r \in \operatorname{Span}(\mathcal{S}) \text{ as } \alpha \beta_j + \gamma_j \in \mathbb{R}\ \forall j.$$

EXS (1) $\mathrm{Span}\left\{\begin{pmatrix}1\\3\end{pmatrix}\right\}$ is a line thru $\begin{pmatrix}0\\0\end{pmatrix}$ of $\mathbb{R}^2$

(2) $\mathrm{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ is xy-plane in $\mathbb{R}^3$

(3) $S = \{ A \in \mathbb{R}^{n\times n} \mid A_{ij} = 0 \text{ for } i \neq j \}$ DIAGONAL MATRICES

is a VSS of $\mathbb{R}^{n\times n}$ as:

Let $\alpha \in \mathbb{R}$, $A, B \in S$. Then for $i \neq j$

$$(\alpha A + B)_{ij} = \alpha A_{ij} + B_{ij} = \alpha\, 0 + 0 = 0$$

So $\alpha A + B \in S$ too.

(4) $S = \left\{ \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 \mid y = x + 1 \right\}$ is NOT a VSS of $\mathbb{R}^2$ as

$(0,1) \in S$ but $(0,2) = 2(0,1) \notin S$.

— GIVE SIMPLE COUNTEREXAMPLE!!

(5) Let $P_n = \mathrm{Span}\{1, x, x^2, \dots, x^n\}$ $\boxed{f_k(x) = x^k \in C^\infty}$

$= \{ \alpha_0 + \alpha_1 x + \dots + \alpha_n x^n \mid \alpha_j \in \mathbb{R} \}$

$=$ Polys of degree $\leq n$.

Then

P_n is a VSS of $C^\infty(\mathbb{R}, \mathbb{R})$.

(6) Let $S = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle/ a, b, d \in \mathbb{R} \right\} \subseteq \mathbb{R}^{2\times 2}$. (3.B)

Lets check S is a subspace of $\mathbb{R}^{2\times2}$.

<u>ALT</u> $S = \{ A \in \mathbb{R}^{2\times2} / A_{21} = 0 \}$

(a) $0 \in S$.

(b) Let $\alpha \in \mathbb{R}$, $A, B \in S$

Then $(\alpha A + B)_{21} = \alpha A_{21} + B_{21} = \alpha 0 + 0 = 0$.

So $\alpha A + B \in S$.

So S is a subspace

[M, 3.3, 3.5]

DEF 6 Let V, W be V.S's. $F: V \to W$ is a **LINEAR TRANSFORMATION** if $\forall \alpha \in \mathbb{R}, \forall \vec{v}_1, \vec{v}_2 \in V$:

$$F(\underbrace{\alpha \vec{v}_1 + \vec{v}_2}_{\text{IN } V}) = \underbrace{\alpha F(\vec{v}_1) + F(\vec{v}_2)}_{\text{IN } W}$$

EX (1) Let A be $m \times n$ matrix. Define

$$F: \mathbb{R}^n \to \mathbb{R}^m \quad \text{by}$$

$$F(\vec{x}) = A\vec{x} \quad \text{where}$$

$$(A\vec{x})_i := \sum_{j=1}^{n} A_{ij} x_j \qquad i = 1, \dots, m$$

(2) **SPECIAL CASE $m = 1$.**

Let $\vec{a}$ be $n \times 1$ col vector, $A = \vec{a}^T$ $1 \times n$.
Then

$$F: \mathbb{R}^n \to \mathbb{R}$$

$$F(\vec{x}) = \vec{a}^T \vec{x} = \sum_{j=1}^{n} a_j x_j = \vec{a} \cdot \vec{x}.$$

PF of (1)

$$F(\alpha \vec{x} + \vec{y}) = \sum_{j=1}^{n} A_{ij} (\alpha \vec{x} + \vec{y})_j = \sum_{j=1}^{n} A_{ij} (\alpha x_j + y_j)$$

$$= \alpha \sum_{j=1}^{n} A_{ij} x_j + \sum_{j=1}^{n} A_{ij} y_j = \alpha A\vec{x} + A\vec{y}$$

$$= \alpha F(\vec{x}) + F(\vec{y}) \quad \blacksquare$$

CONCRETE EXS

① $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a_1 x_1 + a_2 x_2 \qquad \text{(DOT PRODUCT)}$$

LINEARITY

ⓐ $$F\left(\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = F\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = a_1 \alpha x_1 + a_2 \alpha x_2$$

$$= \alpha\,(a_1 x_1 + a_2 x_2) = \alpha\, F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

ⓑ $$F\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = F\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = a_1 (x_1 + y_1) + a_2 (x_2 + y_2)$$

$$= (a_1 x_1 + a_2 x_2) + (a_1 y_1 + a_2 y_2)$$

$$= F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + F\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

② $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} x_1 + a_{12} x_2 \\ a_{21} x_1 + a_{22} x_2 \end{pmatrix} = \begin{pmatrix} F_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ F_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{pmatrix}$$

where $F_j : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is as in ①.

Each F_j is linear. So

$$F(\alpha \vec{x} + \vec{y}) = \begin{pmatrix} F_1(\alpha \vec{x} + \vec{y}) \\ F_2(\alpha \vec{x} + \vec{y}) \end{pmatrix} = \begin{pmatrix} \alpha\, F_1(\vec{x}) + F_1(\vec{y}) \\ \alpha\, F_2(\vec{x}) + F_2(\vec{y}) \end{pmatrix}$$

$$= \alpha \begin{pmatrix} F_1(\vec{x}) \\ F_2(\vec{x}) \end{pmatrix} + \begin{pmatrix} F_1(\vec{y}) \\ F_2(\vec{y}) \end{pmatrix} = \alpha\, F(\vec{x}) + F(\vec{y})$$

CLAIM

$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ xy \end{pmatrix}$ $\qquad f : \mathbb{R}^2 \to \mathbb{R}^2 \rightsquigarrow$ NOT LINEAR

5B

PF SIMPLE COUNTEREXAMPLE

$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad f\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

So $f\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 4\, f\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

If f were linear we would need

$$f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = f\left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = 2\, f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

So f cannot be linear.

3. $\frac{d}{dx} : C^{\infty}(\mathbb{R}, \mathbb{R}) \longrightarrow C^{\infty}(\mathbb{R}, \mathbb{R})$ is a L.T.

4. $\text{Trace} : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$ is a L.T. where

$$\text{Trace}(A) = \sum_{i=1}^{n} A_{ii}$$

FROM COMPOSITION OF LTs TO MATRIX MULTIPLICATION

THM 7 Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a L.T.
Then $\exists$ $m \times n$ A so that
$$F(\vec{x}) = A\vec{x}.$$

PROOF

Let $\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$ $\leftarrow$ jTH ROW for $j = 1, \ldots, n$

$\vec{f}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^m$ for $i = 1, \ldots, m$

If $\vec{x} = \begin{pmatrix} x_1 \\ | \\ x_n \end{pmatrix} \in \mathbb{R}^n$ Then $\vec{x} = \sum_{j=1}^{n} x_j \vec{e}_j$

If $\vec{y} = \begin{pmatrix} y_1 \\ | \\ y_m \end{pmatrix} \in \mathbb{R}^m$ Then $\vec{y} = \sum_{i=1}^{m} y_i \vec{f}_i$

Choosing $\vec{y} = F(\vec{e}_j)$ $\exists A_{ij}$ so that

$$\boxed{F(\vec{e}_j) = \sum_{i=1}^{m} A_{ij} \vec{f}_i} \quad (1) \quad \boxed{\text{THIS DEFINES } m \times n \ A}$$

Then

$$\begin{aligned}
F(\vec{x}) &= F\left(\sum_{j=1}^{n} x_j \vec{e}_j\right) \\
&= \sum_{j=1}^{n} x_j F(\vec{e}_j) && \text{as } F \text{ L.T.} \\
&= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} A_{ij} \vec{f}_i\right) && \text{by } (1) \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} x_j\right) \vec{f}_i && \text{by VS Axioms} \\
&\overset{(*)}{=} \sum_{i=1}^{m} (A\vec{x})_i \vec{f}_i && \text{by def}^n\ A\vec{x}. \\
&= A\vec{x}
\end{aligned}$$

<u>NOTE</u> (*) TELLS US WHAT DEFN OF $A\vec{x}$ <u>HAS</u> TO BE!!

COMPOSITIONS

PROP 8

Let $F: \mathbb{R}^n \to \mathbb{R}^m$, $G: \mathbb{R}^m \to \mathbb{R}^p$ be LTs

Then

$$H = G \circ F : \mathbb{R}^n \to \mathbb{R}^p \text{ is a L.T.}$$

PF You do it.

So by THM 7

$$\vec{y} = F(\vec{x}) = A\vec{x} \quad \text{for } m \times n \ A$$
$$\vec{z} = G(\vec{y}) = B\vec{y} \quad \text{for } p \times m \ B$$

$$\vec{z} = H(\vec{x}) = C\vec{x} \quad \text{for } p \times n \ C.$$

THM 9 $\quad C = BA$

PF

$$C\vec{x} = H(\vec{x}) = G(F(\vec{x})) = G(A\vec{x}) = B(A\vec{x}) = (BA)\vec{x}.$$

Since this holds $\forall \vec{x} \in \mathbb{R}^n$, $C = BA$ must hold.

by

PROP 10 Suppose M is $m \times n$ and $M\vec{x} = \vec{0} \ \forall \vec{x} \in \mathbb{R}^n$

Then $M = 0$ must hold

PF You do it!

ALT If you know $M\vec{x} \ \forall \vec{x} \in \mathbb{R}^n$ Then you know M.

$$\begin{array}{ccccc}
\mathbb{R}^n & \longrightarrow & \mathbb{R}^m & \longrightarrow & \mathbb{R}^p \\
\vec{x} & & \vec{y} & & \vec{z} \\
 & & A\vec{x} & & (BA)\vec{x} \\
j & & k & & i
\end{array}$$

CLAIM 11

Using
$$(M\vec{u})_i = \sum_{j=1}^{n} M_{ij} u_j$$
and
$$(BA)\vec{x} := B(A\vec{x})$$
we can derive formula for matrix mult^n!!
$$(BA)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj} \qquad (*)$$

PF
$$\sum_{j=1}^{n} (BA)_{ij} x_j = \left[(BA)\vec{x}\right]_i$$
$$= \left[B(A\vec{x})\right]_i$$
$$= \sum_{k=1}^{m} B_{ik} (A\vec{x})_k$$
$$= \sum_{k=1}^{m} B_{ik} \sum_{j=1}^{n} A_{kj} x_j$$
$$= \sum_{j=1}^{n} \left(\sum_{k=1}^{m} B_{ik} A_{kj} \right) x_j \qquad \forall \vec{x} \in \mathbb{R}^n$$

~~Hence~~ yields $(*)$.

$\blacksquare$

TWO WAYS TO THINK ABOUT $C = BA$

(I) Let B_{i*} = Row i of B
A_{*j} = Col j of A

Then

$$C_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj} = B_{i*} \cdot A_{*j}$$

$(BA)_{ij}$ = DOT PRODUCT OF ROW i of B WITH COL j of A.

(II) Since

$$(BA)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}$$

$$\boxed{(BA)_{*j} = \sum_{k=1}^{m} A_{kj} B_{*k}}$$

COL j of BA = LIN^R COMB^N of Cols of B where coefficients are entries in col j of A.

<u>TO HELP RECALL:</u>

A is $m \times n$
B is $p \times m$
BA is $p \times n$

<u>SPECIAL CASE:</u>

$A = [\vec{v}_1, \ldots, \vec{v}_n]$ $\quad \vec{x} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

$\vec{b} = A\vec{x} = \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n$

So B_{*k} are $(BA)_{*j}$ are both $p \times 1$.

BLOCK MATRIX MULTIPLICATION

EX

$$A_{4\times4} = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) = \begin{pmatrix} C_{2\times2} & I_{2\times2} \\ 0_{2\times2} & J_{2\times2} \end{pmatrix}$$

$$B_{4\times2} = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 5 & 6 \\ 7 & 8 \end{array}\right) = \begin{pmatrix} 0_{2\times2} \\ D_{2\times2} \end{pmatrix}$$ are examples of block matrices

You multiply block matrices just like you multiply matrices :

$$AB = \begin{pmatrix} C & I \\ 0 & J \end{pmatrix} \begin{pmatrix} 0 \\ D \end{pmatrix}$$ Blocks are Conformable

$$= \begin{pmatrix} C0 + ID \\ 00 + JD \end{pmatrix} = \begin{pmatrix} D \\ JD \end{pmatrix} = \left(\begin{array}{cc} 5 & 6 \\ 7 & 8 \\ \hline -7 & -8 \\ 5 & 6 \end{array}\right)$$

If you ignore blocking and multiply AB as usual you will get same answer.

[M, 3.7]

MATRIX INVERSION

DEF 12 An $n \times n$ matrix A is INVERTIBLE if $\exists$ $n \times n$ B so that

$$AB = I = BA$$

PROP 13

1. If A is invertible then B is unique.
   We write
   $$B = A^{-1}.$$

2. ~~St~~ If A is invertible then sol^n of
   $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$.

3. $(A^{-1})^{-1} = A$

4. $(AB)^{-1} = B^{-1}A^{-1}$

5. $(A^{-1})^{T} = (A^{T})^{-1}$ where $(A^{T})_{ij} = A_{ji}$.

PROOFS OF 4 + 5

4 Show $B^{-1}A^{-1}$ satisfies def^n of inverse of AB:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and $(B^{-1}A^{-1})(AB) = \cdots = I$

So by 1 $(AB)^{-1} = B^{-1}A^{-1}$!

5 First show

$$(AB)^T = B^T A^T$$

Pf $(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{m} A_{jk} B_{ki}$

$$= \sum_{k=1}^{m} B^T_{ik} A^T_{kj} = (B^T A^T)_{ij} \checkmark$$

Then show $(A^{-1})^T$ satisfies defn of inverse of A^T
Well

$$A^T (A^{-1})^T = (A^{-1} A)^T = I^T = I \checkmark$$

etc

$\square$