Math 4355
Matlab Homework #C

You may work either solo or in a group of two. If you work in a group of two, each student must upload their own report into eLearning and you must list both names at the top of the report and briefly state who did what.

Turn in a *single pdf file* with your answers to the questions.

Read background material on *Oscillatory Systems: Eigenvalues* on the next two pages. That material will help you understand how to set up a coupled system of second order ODEs to model a spring mass system. The second order ODEs in (2.26) can be converted to a first order ODE system of the form $\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}$ where $\mathbf{u} = \mathbf{u}(t) \in \mathbb{R}^4$ has components $\mathbf{u} = [X_1, Y_1, X_2, Y_2]^T$ where $Y_j := X_j'$ is the time derivative of X_j. Here A is a 4×4 matrix. Similarly, a spring mass system with N masses will result in a first order ODE with $\mathbf{u} = \mathbf{u}(t) \in \mathbb{R}^{2N}$.
Do Exercises 5.1.22 and 5.1.23 (scanned from a different book). The material summarized above will help you derive first order systems of ODE's that model these two spring mass systems. *Make sure you comment on whether the results you obtain are physically reasonable or not.* If they do not look reasonable maybe you have a bug!
These two exercises refer to Exercises 5.1.19 and 5.1.20 and Example 1.2.10 scans of which are included here for your convenience.

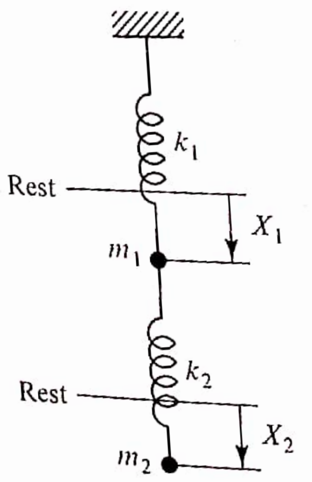
9. Suppose in Problem 8 that the flexibilities k_i all equal 1 and that the applied forces are $f_1 = 0$, $f_2 = 1$, $f_3 = -10$, $f_4 = 1$, and $f_5 = -10$. Use MATLAB or similar software to solve the five linear equations (2.24) in the five unknowns **d** so as to obtain the displacements **d**.

2.5 OSCILLATORY SYSTEMS: EIGENVALUES

Many phenomena in various application areas exhibit oscillations: Airplane wings, bridges, and tall buildings oscillate in the wind; the economy oscillates (between inflation and deflation, for example); and so on. It is obviously important to understand the qualitative behavior of these oscillations: Will the wing fall off, the bridge collapse, the building topple, the economy crumble? The study of models of these phenomena usually leads to matrix problems in which we need to discover when certain matrices depending on parameters will be singular.

Two Masses Suspended and Coupled by Springs

As an illustration of oscillatory phenomena, we consider the motion of two masses coupled by one spring and suspended from the ceiling by another, as in the diagram below; the springs are assumed to have negligible weights. The two weights have masses m_1 and m_2. The two horizontal lines marked "Rest" indicate the position of the masses at rest, that is, where the restoring forces of the springs and the force of gravity are in perfect balance.



We want to model the vertical oscillations of the masses as time passes when they are not at rest. To do so, we introduce two functions of time t: $X_1(t)$ is the downward displacement at time t of the first mass from its rest position, and $X_2(t)$ is the downward displacement at time t of the second mass from its rest position. Then the forces acting on the first mass are (1) the upward force exerted by the first spring, which has been stretched $X_1(t)$ beyond equilibrium; and (2) the downward force exerted by the second spring, which has been stretched by $X_2(t) - X_1(t)$ beyond equilibrium; note that the force of gravity has been accounted for in the rest positions of the masses. The force on the second mass is the upward force exerted by the second spring.

We suppose that the displacements involved are small enough that Hooke's law is valid for the springs: The restoring force for each spring is equal to a constant k_i times the amount by which the spring is stretched. Recalling Newton's law that force equals mass times resulting acceleration and recalling that acceleration is the second derivative of displacement $X_i(t)$, we obtain the mathematical model of our system:

(2.26)
$$\begin{aligned} m_1X_1'' &= -k_1X_1 + k_2(X_2 - X_1) \\ m_2X_2'' &= -k_2(X_2 - X_1), \end{aligned}$$

where the primes denote differentiation with respect to t.

Our mathematical problem is to find functions X_1 and X_2 that satisfy (2.26). Experience and intuition tell us that such systems should be oscillatory: Both X_1 and X_2 should oscillate between positive and negative values much like the trigonometric functions sine and cosine. Therefore, we decide to see whether we can find solutions of (2.26) by using sines and cosines; more precisely, we seek solutions of the form

(2.27)
$$\begin{aligned} X_1(t) &= \xi_1 \sin \omega t + \eta_1 \cos \omega t \\ X_2(t) &= \xi_2 \sin \omega t + \eta_2 \cos \omega t, \end{aligned}$$

where $\xi_1, \eta_1, \xi_2, \eta_2$, and ω are constants to be determined in order that X_1 and X_2 solve (2.26).

If we substitute into (2.26) the expressions for X_1 and X_2 in (2.27), differentiate as required, and collect terms, we obtain

$$\{-m_1\omega^2\xi_1 + k_1\xi_1 + k_2\xi_1 - k_2\xi_2\} \sin \omega t + \{-m_1\omega^2\eta_1 + k_1\eta_1 + k_2\eta_1 - k_2\eta_2\} \cos \omega t = 0 \qquad \text{for all } t$$

$$\{-m_2\omega^2\xi_2 + k_2\xi_2 - k_2\xi_1\} \sin \omega t + \{-m_2\omega^2\eta_2 + k_2\eta_2 - k_2\eta_1\} \cos \omega t = 0 \qquad \text{for all } t.$$

Now, the only way that $A \sin \omega t + B \cos \omega t$ can equal zero for all t is for $A = B = 0$; this means that the expressions above inside braces $\{\cdot\}$ must all equal zero:

(2.28)
$$\begin{aligned} (-m_1\omega^2 + k_1 + k_2)\xi_1 + \qquad (-k_2)\xi_2 &= 0 \\ (-k_2)\xi_1 + (-m_2\omega^2 + k_2)\xi_2 &= 0 \end{aligned}$$

currents to be within (i) ten p... (ii) one percent.

Exercise 5.1.22 Consider a cart attached to a wall by a spring, as shown in Figure 5.5. At

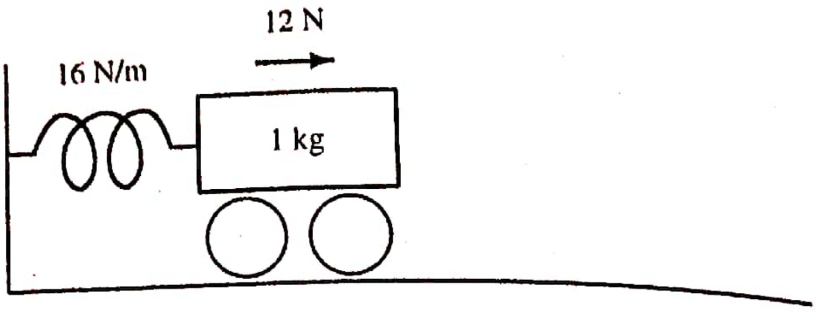


Figure 5.5 Solve for the motion of the cart.

time zero the cart is at rest at its equilibrium position $x = 0$. At that moment a steady force of 12 newtons is applied, pushing the cart to the right. Assume that the rolling friction is $-k\,\dot{x}(t)$ newtons, where $k \geq 0$. Do parts (a) through (d) by hand.

(a) Set up a system of two first-order differential equations of the form $\dot{x} = Ax - b$ for $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$.

(b) Find the steady-state solution of the differential equation.

(c) Find the characteristic equation of A and solve it by the quadratic formula to obtain an expression (involving k) for the eigenvalues of A.

(d) There is a *critical value* of k at which the eigenvalues of A change from real to complex. Find this critical value.

(e) Using MATLAB, solve the initial value problem for the cases (i) $k = 2$, (ii) $k = 6$, (iii) $k = 10$, and (iv) $k = 14$. Rather than reporting your solutions, simply plot $x_1(t)$ for $0 \leq t \leq 3$ for each of your four solutions on a single set of axes. (Do not overlook the help given in Exercises 5.1.19 and 5.1.20.) Comment on your plots (e.g. rate of decay to steady state, presence or absence of oscillations).

(f) What happens when $k = 0$?

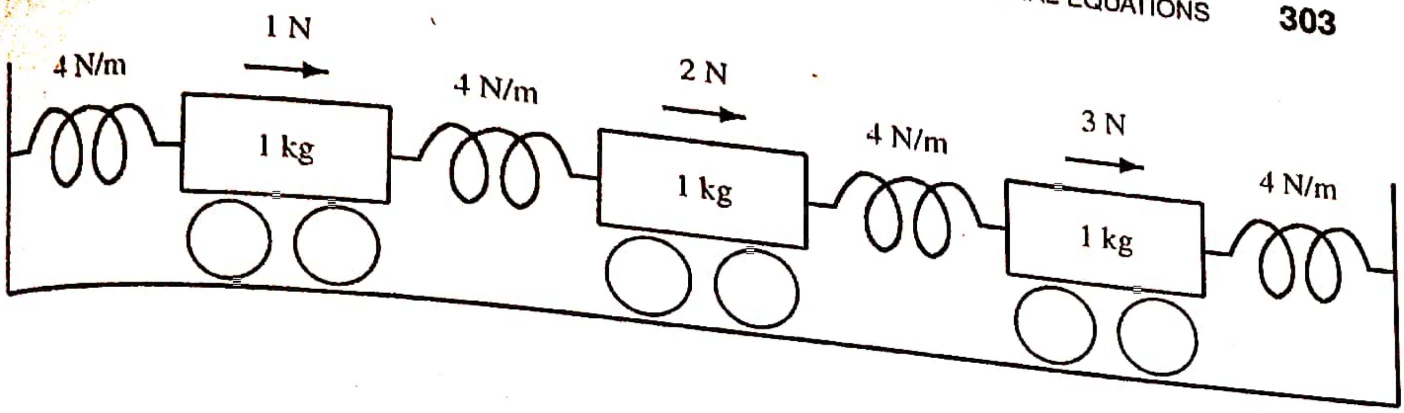


Figure 5.6 Solve for the motion of the carts.

Exercise 5.1.23 Consider a system of three carts attached by springs, as shown in Figure 5.6. The carts are initially at rest. At time zero the indicated forces are applied, causing the carts to move toward a new equilibrium. Let x_1, x_2, and x_3 denote the displacements of the carts, and let x_4, x_5, and x_6 denote their respective velocities. Suppose the coefficients of rolling friction of the three carts are k_1, k_2, and k_3, respectively.

(a) Apply Newton's second law to each cart to obtain a system of three second-order differential equations for the displacements x_1, x_2, and x_3. You may find it useful to review Example 1.2.10.

(b) Introducing the velocity variables x_4, x_5, and x_6, rewrite your system as a system of six first-order differential equations. Write your system in the form $\dot{x} = Ax - b$.

(c) Find the steady-state solution of the system.

(d) Solve the initial value problem under each of the conditions listed below. In each case plot x_1, x_2, and x_3 on a single set of axes for $0 \leq t \leq 20$, and comment on the plot.

(1) $k_1 = 1$, $k_2 = 0$, and $k_3 = 0$.

(2) $k_1 = 1$, $k_2 = 8$, and $k_3 = 8$.

(3) $k_1 = 8$, $k_2 = 8$, and $k_3 = 8$.

□

Additional Exercises

...cise 5.1.19 Suppose A is nonsingular and has linearly independent eigenvectors $v_1, \ldots,$ v_n. Let V be the nonsingular $n \times n$ matrix whose columns are $v_1, \ldots, v_n$.

(a) Show that (5.1.8) can be rewritten as $x(t) = Ve^{\Lambda t}c$, where c is a column vector, Λt is the diagonal matrix $\text{diag}\{\lambda_1 t, \ldots, \lambda_n t\}$, and $e^{\Lambda t}$ is its *matrix exponential*: $e^{\Lambda t} = \text{diag}\{e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\}$.

(b) Show that the general solution of $\dot{x} = Ax - b$ has the form $x(t) = z + Ve^{\Lambda t}c$, where z satisfies $Az = b$.

(c) To solve the initial value problem $\dot{x} = Ax - b$ with initial condition $x(0) = \hat{x}$, we need to solve for the constants in the vector c. Show that c can be obtained by solving the system $Vc = \hat{x} - z$. Since V is nonsingular, this system has a unique solution.

□

Exercise 5.1.20 Using MATLAB, work out the details of Example 5.1.13. The MATLAB command `[V,D]=eig(A)` returns (if possible) a matrix V whose columns are linearly independent eigenvectors of A and a diagonal matrix D whose main diagonal entries are the eigenvalues of A. Thus V and D are the same as the matrices V and Λ of the previous exercise. You may find the results of the previous exercise helpful as you work through this exercise. Here are some sample plot commands:

```
t = 0:.02:1;
x = z*ones(size(t));
for j=1:2; x = x + V(:,j)*c(j)*exp(t*D(j,j)); end
plot(t,x(1,:),t,x(2,:),'--')
title('Loop Currents')
xlabel('time in seconds')
ylabel('current in amperes')
print
```

Remember that for more information on the usage of these commands, you can type `help plot`, `help print`, etc., or look in MATLAB's help browser. □

some other means.

□

It is easy to imagine much larger circuits with many loops. See, for example, Exercise 1.2.19. Then imagine something much larger. If a circuit has, say, 100 loops, then it will have 100 equations in 100 unknowns.

Simple Mass-Spring Systems

In Figure 1.3 a steady force of 2 newtons is applied to a cart, pushing it to the right and stretching the spring, which is a linear spring with a spring constant (stiffness) 4 newtons/meter. How far will the cart move before stopping at a new equilibrium position? Here we are not studying the dynamics, that is, how the cart gets to its new equilibrium. For that we would need to know the mass of the cart and the frictional forces in the system. Since we are asking only for the new equilibrium position, it suffices to know the stiffness of the spring.

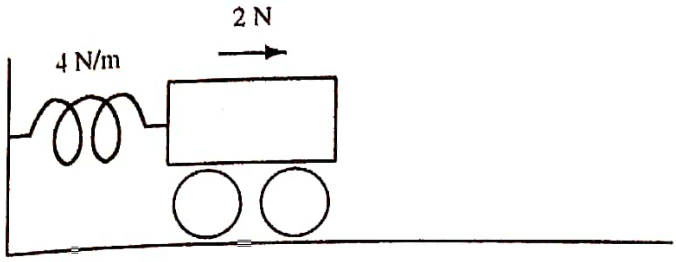


Figure 1.3 Single cart and spring

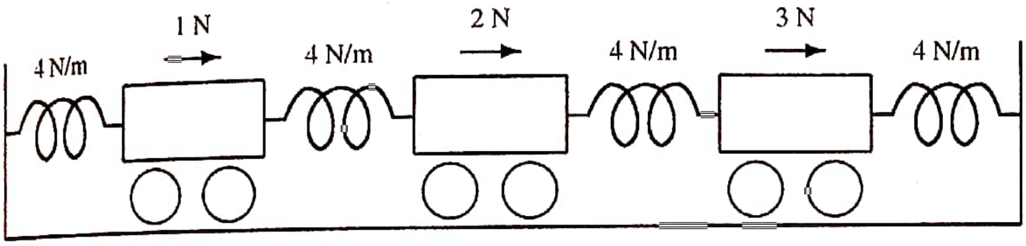


Figure 1.4 System of three carts

The new equilibrium will be at the point at which the rightward force of 2 newtons is exactly balanced by the leftward force applied by the spring. In other words, the equilibrium position is the one at which the sum of the forces on the cart is zero. Let x denote the (yet unknown) amount by which the cart moves to the right. Then the restoring force of the spring is -4 newtons/meter $\times$ x meters = $-4x$ newtons. It is negative because it pulls the cart leftward. The equilibrium occurs when $-4x+2=0$. Solving this system of one equation in one unknown, we find that $x = 0.5$ meter.

Example 1.2.10 Now suppose we have three masses attached by springs as shown in Figure 1.4. Let x_1, x_2, and x_3 denote the amount by which carts 1, 2, and 3, respectively, move when the forces are applied. For each cart the new equilibrium position is that point at which the sum of the forces on the cart is zero. Consider the second cart, for example. An external force of two newtons is applied, and there is the leftward force of the spring to the left, and the rightward force of the spring to the right. The amount by which the spring on the left is stretched is $x_2 - x_1$ meters. It therefore exerts a force -4 newtons/meter $\times$ $(x_2 - x_1)$ meters = $-4(x_2 - x_1)$ newtons on the second cart. Similarly the spring on the right applies a force of $+4(x_3 - x_2)$ newtons. Thus the equilibrium equation for the second cart is

$$-4(x_2 - x_1) + 4(x_3 - x_2) + 2 = 0$$

or

$$-4x_1 + 8x_2 - 4x_3 = 2.$$

Similar equations apply to carts 1 and 3. Thus we obtain a system of three linear equations in three unknowns, which we can write as a matrix equation

$$\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Entering the matrix A and vector b into MATLAB, and using the command `x = A\b` (or simply solving the system by hand) we find that

$$x = \begin{bmatrix} 0.625 \\ 1.000 \\ 0.875 \end{bmatrix}.$$

Thus the first cart is displaced to the right by a distance of 0.625 meters, for example.

The coefficient matrix A is called a *stiffness matrix*, because the values of its nonzero entries are determined by the stiffnesses of the springs. □