Math 4355
Matlab Homework #A

You may work either solo or in a group of two. If you work in a group of two, each student must upload their own report into eLearning and you must list both names at the top of the report and briefly state who did what.

Your task is to write MATLAB functions to compute the four fundamental subspaces and the rank normal form of a matrix. Test your code using the matrices in (**A**) Exercises 3.9.1, (**B**) on the transpose of the matrix in Exercise 3.9.1, and the matrices in (**C**) 4.2.1, and (**D**) 4.4.2 of Meyer.

Turn in a *single pdf file* containing the answers to the conceptual questions posed below, a print out of your code, and of the results you obtain by running the code on the four test matrices above. Make sure your code is commented well enough so that you will understand it in two month's time.

When you generate your output, first provide the answers to each of items 1-6 below for the matrix **A** *above, then repeat for* **B**, **C**, *and* **D**.

Background reading from Meyer: Section 2.2 (especially page 48), Section 3.9 (especially page 136), Exercise 3.9.1 (page 139), Section 4.2 (pages 169-178), Page 199 of Section 4.4.

1. Write a function B = BasisOfRange(A) whose input A is a matrix and whose output B is a matrix whose columns form a basis for the range of A.
   - Recall that the basic (pivot) columns of $\mathbf{A}$ form a basis for $\mathcal{R}(\mathbf{A})$.
   - You can use the built-in matlab function rref to compute the reduced row echelon form of $\mathbf{A}$ and identify the pivot columns.
2. Write a function B = BasisOfRangeOfTranspose(A) whose input A is a matrix and whose output B is a matrix whose columns form a basis for the range of A^T.
   - *In this function you must use the fact that the nonzero rows of any row echelon form of* $\mathbf{A}$ *form a basis for* $\mathcal{R}(\mathbf{A}^T)$*.* Do not simply take the transpose of $\mathbf{A}$ and then use the function you wrote above. The rationale for this is that you want to do row operations just once (on $\mathbf{A}$), rather than doing them first on $\mathbf{A}$ to find a basis for $\mathcal{R}(\mathbf{A})$ and then again on $\mathbf{A}^T$ to find a basis for $\mathcal{R}(\mathbf{A}^T)$.
3. If you pass a matrix $\mathbf{M}$ to the function BasisOfRangeOfTranspose you will obtain a basis for $\mathcal{R}(\mathbf{M}^T)$. As suggested above, you could also obtain a basis for $\mathcal{R}(\mathbf{M}^T)$ by passing the matrix $\mathbf{M}^T$ to the function BasisOfRange. Are these two bases necessarily the same? Explain.
4. Write a function B = BasisOfNullSpace(A) whose input A is a matrix and whose output B is a matrix whose columns form a basis for the nullspace of A. Use the following method to compute this basis.

- Let $\mathbf{A}$ be an $m \times n$ matrix of rank r. Recall (Meyer page 175) that the general solution, $\mathbf{x} = [x_1, \cdots, x_n]^T$, of $\mathbf{Ax} = \mathbf{0}$ can be expressed in the form

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}. \tag{1}$$

Here $f_1, \cdots, f_{n-r}$ are the indices of the free variables. Furthermore, the matrix

$$\mathbf{H} = [\mathbf{h}_1, \cdots, \mathbf{h}_{n-r}] \tag{2}$$

is an $n \times (n-r)$ matrix whose columns form a basis for $N(\mathbf{A})$.

- To obtain $\mathbf{x} = \mathbf{h}_1$ we need to set $x_{f_1} = 1$ and $x_{f_k} = 0$ for all $k > 1$. Then (2) holds. The remaining entries of the vector $\mathbf{x}$ are the basic variables. Examining the k-th row of the reduced row echelon form on Meyer page 48, you can see how to solve for the k-th basic variable in terms of the free variables.
- Specifically, proceed as follows. Let $b_1, \cdots b_r$ be the indices of the basic variables. So

$$\{f_1, \cdots, f_{n-r}\} \cup \{b_1, \cdots, b_r\} = \{1, \cdots, n\} \tag{3}$$

and

$$\{f_1, \cdots, f_{n-r}\} \cap \{b_1, \cdots, b_r\} = \emptyset. \tag{4}$$

Let $\mathbf{E}$ be the reduced row echelon form of $\mathbf{A}$. Then the entries of the matrix $\mathbf{H}$ are given by

$$\mathbf{H}_{f_i j} = \delta_{ij} \tag{5}$$

and

$$\mathbf{H}_{b_k j} = -(\mathbf{E}_{k f_1}\mathbf{H}_{f_1 j} + \cdots \mathbf{E}_{k f_{n-r}}\mathbf{H}_{f_{n-r} j}). \tag{6}$$

- Explain why (5) and (6) are true in the context of Meyer Example 3.9.1.
- MATLAB has a built in function null that computes the nullspace of a matrix. Verify that your code is correct by comparing the results you obtain using BasisOfNullSpace to null. How else could you check that you have correctly computed the nullspace?

5. Write a function B = BasisOfNullSpaceOfTranspose(A) whose input A is a matrix and whose output B is a matrix whose rows form a basis for the nullspace of A^T.

    - Let $\mathbf{P}$ be the $m \times m$ nonsingular matrix so that $\mathbf{E} = \mathbf{PA}$ is the reduced row echelon form of $\mathbf{A}$.
    - Recall from Meyer pages 176 and 199 that the last $m - r$ rows of $\mathbf{P}$ form a basis for $N(\mathbf{A}^T)$.
    - To compute $\mathbf{P}$ you can use Meyer Exercise 3.9.1(a) which tells us that the reduced row echelon form of $[\mathbf{A}|\mathbf{I}]$ is $[\mathbf{E}|\mathbf{P}]$.

6. Write a function RankNormalForm(A) whose input A is an $m \times n$ matrix and which computes invertible $\mathbf{P}$ and $\mathbf{Q}$ so that $\mathbf{PAQ}$ is in rank normal form

- Use the method suggested by Meyer Exercise 3.9.1.
- Note that in 3.9.1(b), instead of column reducing a matrix you can row reduce its transpose.
- Also, you will need to use the ideas in (b) to find $\mathbf{Q}$ so that $\mathbf{E_A}$ in 3.9.1(c) satisfies

$$\mathbf{E_A Q} = \begin{bmatrix} \mathbf{I}_{r\times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{7}$$

HINTS: Some basic matrix manipulations in Matlab

- $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3; 4 & 5 & 6 \end{bmatrix}$ generates a 2×3 matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$
- $\mathbf{C} = \mathbf{A} * \mathbf{B}$ is matrix multiplication.
- $\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix}$ is the block matrix $\mathbf{C} = [\mathbf{A}|\mathbf{B}]$
- $[1 : n]$ is short for the array [1,2,3,...,n]
- Suppose $\mathbf{A}$ is an $n \times m$ matrix, and let $I \subseteq [1 : n]$ and $J \subseteq [1 : m]$. Then $\mathbf{A}(I, :)$ is the submatrix of $\mathbf{A}$ consisting of the rows with indices in I. Also $\mathbf{A}(:, J)$ is the submatrix of $\mathbf{A}$ consisting of the columns with indices in J. Finally, $\mathbf{A}(I, J)$ is the submatrix of $\mathbf{A}$ whose indices (i, j) satisfy $i \in I$ and $j \in J$.

Here is a list of basic matlab commands you may find useful. Within matlab type `help size` for a description of the command `size`.

- `size`
- `numel`
- `find`
- `eye`
- `transpose`
- `setdiff`