

Stability of Short Pulse Lasers: Modeling, Numerics, and Analysis

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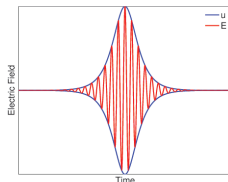
Part I

Computational Modeling of Lasers

Stationary Pulse Solution of NLS Equation

- Electric field of light pulse propagating in **optical fiber**:

$$E(t, x) = 2 \operatorname{Re} \{ \psi(t, x) \exp[i(\beta_0 t - \omega_0 x)] \}$$



- Nonlinear Schrödinger equation** for envelope, ψ :

$$\partial_t \psi = \frac{i}{2} \partial_x^2 \psi + i |\psi|^2 \psi$$

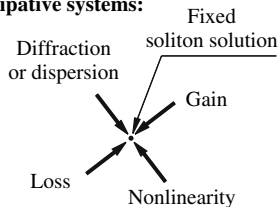
- Famous **Stationary Solution**:

$$\psi(\mathbf{t}, \mathbf{x}) = A \operatorname{sech}(A \mathbf{x}) \exp(i A^2 \mathbf{t} / 2)$$

Short Pulse Fiber Lasers

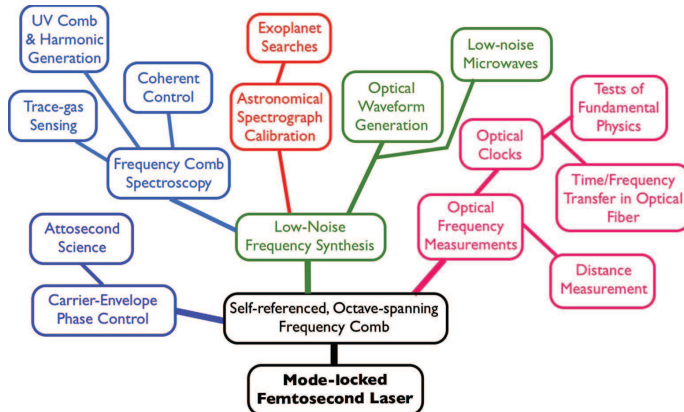
- Generate a train of short, high energy light pulses by balancing
 - **Dispersion** and **nonlinearity**
 - **Gain** and **loss**

Dissipative systems:



- In addition:
 - **Saturable absorber** suppresses noise far from the pulse
 - **Spectral filtering** facilitates pulse formation

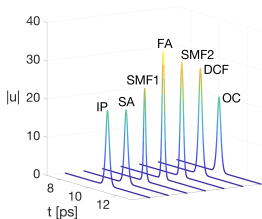
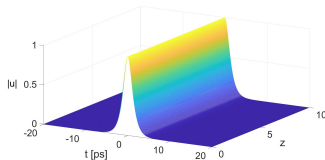
Applications of Short Pulse Fiber Lasers¹



¹Diddams, "The Evolving Frequency Comb", *JOSA B*, **27**, (2010)

Five Generations of Short Pulse Lasers²

- Stationary pulses
 - Soliton laser [1990]
- Periodically stationary (PS) pulses
 - Stretched pulse laser [1995]
 - Similariton laser [2004]
 - Dissipative soliton laser [2006]
 - The Mamyshev oscillator [2017]



Pulse breathing has increased with each successive generation

²Frank Wise: [Guide to Modelocked Lasers](#)

Taxonomy of Laser Models

Averaged Models

- Each physical effect is averaged over one round trip
- Results in a single constant coefficient PDE
- Generate stationary or PS pulses
- Stability analysis only involves **the linearized operator**

Lumped Models

- Concatenation of components each having a different effect
- Component models: input-output devices or PDE's
- Generate PS pulses
- Stability analysis requires **solution of the linearized PDE**

**Need lumped models to capture pulse breathing
in recent generations of mode-locked lasers³**

³Turitsyn et al., *Physics Reports*, (2012)

Cubic-quintic complex Ginzburg-Landau equation:

$$i\partial_t\psi + \left(\frac{D}{2} - i\beta\right)\partial_x^2\psi - i\delta\psi + [\gamma - i\epsilon + (\nu - i\mu)|\psi|^2]|\psi|^2\psi = 0$$

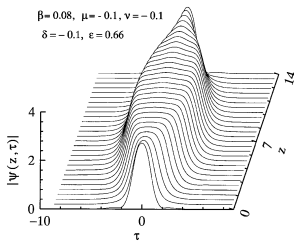
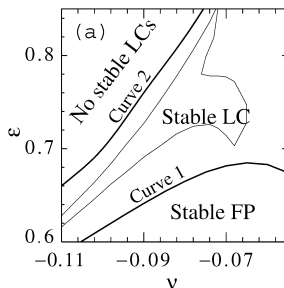


FIG. 1. Plain pulsating soliton of CGLE. The parameters are $D = +1$, $\epsilon = 0.66$, $\delta = -0.1$, $\beta = 0.08$, $\mu = -0.1$, and $\nu = -0.1$.



Numerical evidence for both stable & unstable breathers based on reduced-ODE and PDE solvers

⁴Left figure: Akhmediev et al. PRE (2001)

⁵Right figure: Tsoy and Akhmediev, Phys. Lett. A (2005)

Part II

Monodromy Operators in Lumped Models

Floquet Theory of periodic ODEs

- If we linearize a nonlinear, autonomous ODE

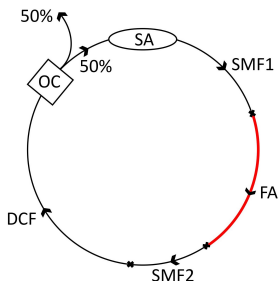
$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \mathbf{F}(\mathbf{x}) && \text{with } \mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n, \\ \mathbf{x}(t_0) &= \mathbf{x}_0,\end{aligned}$$

about a T -periodic solution $\mathbf{x}(t) = [x_1(t), \dots, x_n(t)]^T$ we obtain

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}(t)\mathbf{y}, \quad \text{with } T\text{-periodic } \mathbf{A}(t) = D\mathbf{F}_{\mathbf{x}(t)} \in \mathbb{R}^{n \times n}.$$

- **Monodromy matrix** $\mathbf{M}$ is the solution operator: $\mathbf{y}(T) = \mathbf{M}\mathbf{y}(0)$.
- $\mathbf{y}(nT) = \mathbf{M}^n\mathbf{y}_0$
- **Floquet Stability (Basic Idea):**
 - ① Perturbations $\mathbf{y}$ of $\mathbf{x}$ decay if all eigenvalues of $\mathbf{M}$ satisfy $|\lambda| \leq 1$.
 - ② Otherwise $\mathbf{y}$ can grow without bound.

Model of experimental stretched pulse laser⁶



- **Roundtrip Operator:**

$$\mathcal{R} = \mathcal{P}^{\text{OC}} \circ \mathcal{P}^{\text{DCF}} \circ \mathcal{P}^{\text{SMF2}} \circ \mathcal{P}^{\text{FA}} \circ \mathcal{P}^{\text{SMF1}} \circ \mathcal{P}^{\text{SA}}$$

- $\psi = \psi(x)$ is **periodically stationary** if $\exists \theta$:

$$\mathcal{R}(\psi) = e^{i\theta} \psi.$$

- **Monodromy Operator:**

$$\mathcal{M}\mathbf{u} = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{R}(\psi + \epsilon \mathbf{u}) - \mathcal{R}(\psi)}{\epsilon}$$

$$\mathcal{M} = \mathcal{U}^{\text{OC}} \circ \mathcal{U}^{\text{DCF}} \circ \mathcal{U}^{\text{SMF2}} \circ \mathcal{U}^{\text{FA}} \circ \mathcal{U}^{\text{SMF1}} \circ \mathcal{U}^{\text{SA}}$$

⁶Kim et al, *JSTQE*, **20**, [2014]

- Fiber Amplifier [Nonlocal in x]:**

$$\psi_{\text{out}} = \mathcal{P}^{\text{FA}} \psi_{\text{in}} = \psi(\text{L}_{\text{FA}}, \cdot)$$

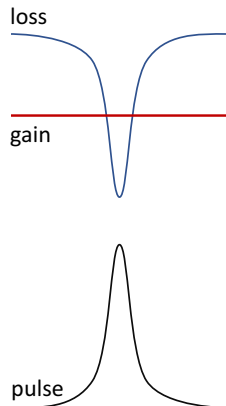
$$\partial_t \psi = \left[\frac{g(\psi)}{2} \left(1 + \frac{1}{\Omega_g^2} \partial_x^2 \right) - \frac{i}{2} \beta \partial_x^2 + i \gamma |\psi|^2 \right] \psi,$$

$$g(\psi) = \frac{g_0}{1 + E(\psi)/E_{\text{sat}}}$$

$$E(t) = \int_{\mathbb{R}} |\psi(t, x)|^2 dx$$

- Saturable Absorber:**

$$\psi_{\text{out}} = \mathcal{P}^{\text{SA}}(\psi_{\text{in}}) = \left(1 - \frac{\ell_0}{1 + |\psi_{\text{in}}|^2/P_{\text{sat}}} \right) \psi_{\text{in}}$$



Linearization of Model Equations

- We reformulate the nonlinear PDE's as systems for

$$\boldsymbol{\psi} = [\text{Re}(\psi) \quad \text{Im}(\psi)]^T.$$

- Linearized solution operator in FA is $\mathbf{u}_{\text{out}} = \mathcal{U}^{\text{FA}} \mathbf{u}_{\text{in}} = \mathbf{u}(L_{\text{FA}}, \cdot)$:

$$\begin{aligned} \partial_t \mathbf{u} &= [g(\boldsymbol{\psi})\mathcal{K} + \mathcal{L} + \mathcal{M}_1(\boldsymbol{\psi}) + \mathcal{M}_2(\boldsymbol{\psi}) + \mathcal{P}(\boldsymbol{\psi})] \mathbf{u}, \\ \mathbf{u}(0, \cdot) &= \mathbf{u}_{\text{in}}, \quad \text{where} \end{aligned}$$

$$\mathcal{K} = \frac{1}{2} \left(1 + \frac{1}{\Omega_g^2} \partial_x^2 \right), \quad \mathcal{L} = -\frac{\beta}{2} \mathbf{J} \partial_x^2,$$

$$\mathcal{M}_1(\boldsymbol{\psi}) = \gamma \|\boldsymbol{\psi}\|^2 \mathbf{J}, \quad \mathcal{M}_2(\boldsymbol{\psi}) = 2\gamma \mathbf{J} \boldsymbol{\psi} \boldsymbol{\psi}^T,$$

and $\mathcal{P}(\boldsymbol{\psi})$ is the **nonlocal operator**

$$\mathcal{P}(\boldsymbol{\psi}) \mathbf{u} = -\frac{g^2(\boldsymbol{\psi})}{g_0 E_{\text{sat}}} \left[\left(1 + \frac{1}{\Omega_g^2} \partial_x^2 \right) \boldsymbol{\psi} \right] \int_{-\infty}^{\infty} \boldsymbol{\psi}^T(x) \mathbf{u}(x) dx.$$

$$\mathcal{M} : L^2(\mathbb{R}, \mathbb{R}^2) \rightarrow L^2(\mathbb{R}, \mathbb{R}^2)$$

Two Stage Method

- **Stage 1:** Propagate Gaussian pulse over sufficiently many round trips obtain a good initial guess for Stage 2.
- **Stage 2:** Minimize $\tilde{\mathcal{E}}(\psi_0, \theta) := \mathcal{E}(\psi_0, \theta)/E(\psi_0)$, where

$$\mathcal{E}(\psi_0, \theta) = \frac{1}{2} \int_{-\infty}^{\infty} |\mathcal{R}(\psi_0(t)) - e^{i\theta} \psi_0(t)|^2 dt,$$

using a **gradient based optimization** method.

We compute the gradient of $\mathcal{E}$ by adapting the adjoint state method of Ambrose and Wilkening for finding periodic solutions of PDE's.⁷

⁷Ambrose and Wilkening (2012)

Existence Theorem for Monodromy Operator

Assuming ψ actually exists, what can we say about $\mathcal{M}$?

Theorem

Under some reasonable boundedness and regularity hypotheses on ψ , $\exists!$ monodromy operator, $\mathcal{M}$:

- (a) $\mathcal{M} \in \mathcal{B}(L^2(\mathbb{R}, \mathbb{C}^2))$;
- (b) $\mathcal{M}(H^2(\mathbb{R}, \mathbb{C}^2)) \subset H^2(\mathbb{R}, \mathbb{C}^2)$;
- (c) *For each $\mathbf{v} \in H^2(\mathbb{R}, \mathbb{C}^2)$, $\mathbf{u} = \mathcal{M}(\mathbf{v})$ is the unique solution after one round trip of the linearization of $\mathcal{R}$ about ψ .*

Proof: Apply analytic evolution semigroup theory⁸ to the operators $\mathcal{U}$.

⁸Pazy, *Semigroups of Linear Operators*; Lunardi, *Analytical Semigroups...*

Stability of PS Pulses

- To obtain a truly **periodic pulse** we modify $\mathcal{R}$ and $\mathcal{M}$:

$$\widetilde{\mathcal{R}} = \mathbf{R}(-\theta) \circ \mathcal{R} \quad \widetilde{\mathcal{M}} = \mathbf{R}(-\theta) \circ \mathcal{M}$$

- $\widetilde{\mathcal{M}}$ is a differential operator on an ∞ -D function space $L^2(\mathbb{R}, \mathbb{C}^2)$
- **Spectrum of $\widetilde{\mathcal{M}}$:**

$$\sigma(\widetilde{\mathcal{M}}) = \sigma_{\text{pt}}(\widetilde{\mathcal{M}}) \cup \sigma_{\text{ess}}(\widetilde{\mathcal{M}})$$

- **Point spectrum** (set of eigenvalues):

$$\sigma_{\text{pt}}(\widetilde{\mathcal{M}}) := \{\lambda \in \mathbb{C} \mid \widetilde{\mathcal{M}}\mathbf{u} = \lambda\mathbf{u} \text{ for some } \mathbf{u} \in L^2(\mathbb{R}, \mathbb{C}^2)\}$$

- Intuition re **Essential spectrum** of $\widetilde{\mathcal{M}}$:

- ① Quantifies growth of noise perturbations far from pulse
- ② $\widetilde{\mathcal{M}}\mathbf{u} = \lambda\mathbf{u}$, but $\mathbf{u}$ decays too slowly to be in $L^2(\mathbb{R}, \mathbb{C}^2)$

ψ is **spectrally stable** iff $\sigma(\widetilde{\mathcal{M}}) \subseteq \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \{1\}$.

Essential Spectrum of $\widetilde{\mathcal{M}}$

Theorem (Essential Spectrum Theorem)

For a lumped model that includes

- fast saturable loss (unsaturated loss ℓ_0) and output coupler (ℓ_{OC})
- saturable gain with gain, $g = g(t)$, and bandwidth Ω_g
- round-trip dispersion, β_{RT} ,

$$\sigma_{\text{ess}}(\widetilde{\mathcal{M}}) = \{\lambda_{\pm}(\omega) : \omega \in \mathbb{R}\} \cup \{0\},$$

$$\begin{aligned} \lambda_{\pm}(\omega) = & \ell_{\text{OC}}(1 - \ell_0) \exp \left\{ \frac{1}{2} \left(1 - \frac{\omega^2}{\Omega_g^2} \right) \int_0^{L_{\text{FA}}} g(\psi(t)) dt \right\} \\ & \times \exp \left\{ \pm i \left(\frac{\beta_{\text{RT}}}{2} \omega^2 - \theta \right) \right\}. \end{aligned}$$

$\sigma_{\text{ess}}(\widetilde{\mathcal{M}})$ is stable if more loss than gain far from pulse.

Eigenvalues of $\widetilde{\mathcal{M}}$

- **Phase invariance eigenfunction:** $\widetilde{\mathcal{M}} \mathbf{u}_{\text{ph}} = \mathbf{u}_{\text{ph}}$ where

$$\mathbf{u}_{\text{ph}}(x) = \mathbf{J} \psi(x).$$

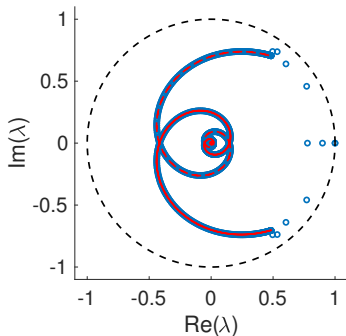
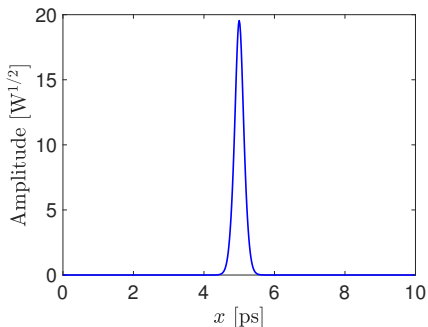
- **Translation invariance eigenfunction:** $\widetilde{\mathcal{M}} \mathbf{u}_{\text{tr}} = \mathbf{u}_{\text{tr}}$ where

$$\mathbf{u}_{\text{tr}}(x) = \psi_x(x).$$

- So, $\lambda = 1$ is an eigenvalue of $\widetilde{\mathcal{M}}$ with multiplicity (at least) two.
- For the rest, approximate $\sigma(\widetilde{\mathcal{M}})$ by eigenvalues of $\mathbf{M} \in \mathbb{R}^{2N \times 2N}$:

$$\mathbf{M}(:, k) = \widetilde{\mathcal{M}}(\mathbf{e}_k).$$

$$g_0 = 6\text{m}^{-1}, E_{\text{sat}} = 200 \text{ pJ}, \Omega_g = 50 \text{ THz}, N = 512$$



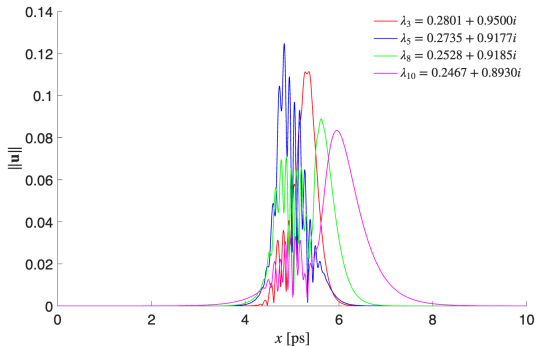
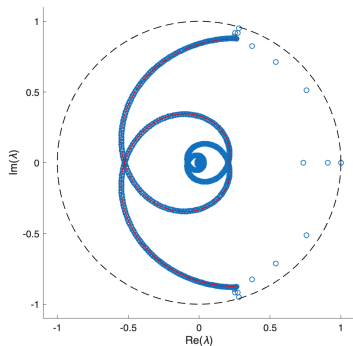
After Stage 1 [Evolution]: $\tilde{\mathcal{E}} = 8 \times 10^{-3}$

After Stage 2 [Optimization]: $\tilde{\mathcal{E}} = 5 \times 10^{-27}$

Errors in $\lambda = 1$: Phase: 10^{-13} & Translation: 4×10^{-11}

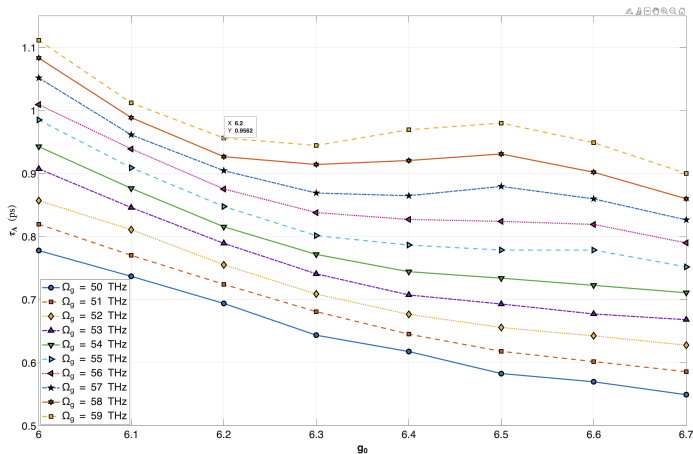
Numerically computed spectrum validates theory

Results with Slow Saturable Absorber



With slow saturable loss, noise can grow in wake of pulse

Stability Boundary Tracking for Slow Saturable Absorber



As amplifier gain decreases, response time of absorber can increase.

Distributing gain across more frequencies increases stability region.

Part III

Birman-Schwinger Operators for Averaged Models

The Big Picture

- Stability of stationary pulses characterized by Evans det, $E(\lambda)$.
- Computation of $E(\lambda)$ requires solution of stiff ODE system
 - Challenging for high dimensional problems
- Can we compute point spectrum for **breathers**??
- Breathers give rise to ∞ -dim ODE system (via Fourier series)
 - Theoretical definition of $E(\lambda)$ may not even be possible!

Alternate Approach:

Via a Fredholm determinant of a trace class operator

Today: **Stationary** Pulses + Low-Dimensions only!

In case of **Complex Ginzburg-Landau Equation**

Cubic-quintic complex Ginzburg-Landau

$$i\partial_t\psi + \left(\frac{D}{2} - i\beta\right)\partial_x^2\psi + (\alpha - i\delta)\psi + [\gamma - i\epsilon + (\nu - i\mu)|\psi|^2] |\psi|^2\psi = 0.$$

Reformulate as a system for $\psi = [\text{Re}(\psi) \quad \text{Im}(\psi)]^T$

$$\partial_t\psi = (\mathbf{B}\partial_x^2 + \mathbf{N}_0 + \mathbf{N}_1|\psi|^2 + \mathbf{N}_2|\psi|^4)\psi,$$

where

$$\mathbf{B} = \begin{bmatrix} \beta & -\frac{D}{2} \\ \frac{D}{2} & \beta \end{bmatrix}, \quad \text{assume } (\beta, D) \neq (0, 0),$$

and

$$\mathbf{N}_0 = \begin{bmatrix} \delta & -\alpha \\ \alpha & \delta \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} \epsilon & -\gamma \\ \gamma & \epsilon \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix}.$$

Linearization of CQ-CGLE

- Linearizing CQ-CGLE about ψ , we obtain

$$\partial_t \mathbf{u} = \mathcal{L} \mathbf{u},$$

$$\mathcal{L} = \mathcal{L}_\infty + \mathbf{M}, \quad \text{with} \quad \mathcal{L}_\infty = \mathbf{B} \partial_x^2 + \mathbf{N}_0,$$

where

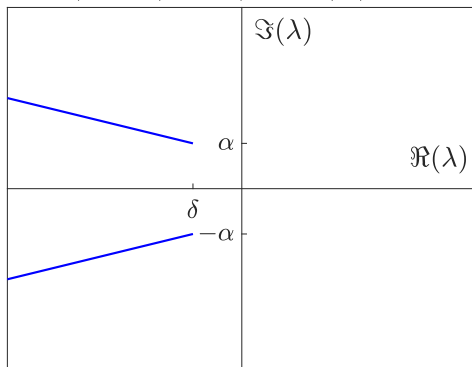
$$\mathbf{M}(x) = \mathbf{N}_1 |\psi|^2 + \mathbf{N}_2 |\psi|^4 + (2\mathbf{N}_1 + 4\mathbf{N}_2 |\psi|^2) \psi \psi^T.$$

- Linear stability of the ψ is determined by

$$\sigma(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}) \cup \sigma_{\text{pt}}(\mathcal{L}).$$

The essential spectrum of $\mathcal{L}$

$$\lambda = (\delta \pm i\alpha) - \mu^2(\beta \pm iD/2), \quad \mu \in \mathbf{R}$$



Stable for $\delta < 0, \beta > 0$

Goal: Find a rigorous way to calculate the point spectrum.

Trace class & Hilbert Schmidt operators; Fredholm dets

$$\mathcal{B}_p(\mathcal{H}) = \{\mathcal{K} : \mathcal{H} \rightarrow \mathcal{H} \text{ is compact} : \sum_{j=1}^{\infty} |\mu_j|^p < \infty\},$$

where $\{\mu_j\}_{j=1}^{\infty}$ are the singular values of $\mathcal{K}$.

- $\mathcal{B}_1(\mathcal{H}) = \text{Trace class} \subsetneq \mathcal{B}_2(\mathcal{H}) = \text{Hilbert Schmidt operators}$
- Every $\mathcal{K} \in \mathcal{B}_2(L^2(\mathbb{R}, \mathbb{C}^k))$ has a kernel $\mathbf{K}(x, y) \in \mathbb{C}^{k \times k}$.
- For $\mathcal{K} \in \mathcal{B}_1(\mathcal{H})$ with $\{\eta_k\}$ an ONB for $\mathcal{H}$:

$$\text{Tr}(\mathcal{K}) := \sum \langle \eta_k, \mathcal{K} \eta_k \rangle = \sum \lambda_i(\mathcal{K}) = \int_{\mathbb{R}} \text{Tr}(\mathbf{K}(x, x)) dx$$

$$\prod_i^{\infty} (1 + \lambda_i) = \sum_{n=0}^{\infty} \sum_{i_1 < \dots < i_n} \lambda_{i_1} \cdots \lambda_{i_n} = \sum_{n=0}^{\infty} \text{Tr}(\wedge^n \mathcal{K}) =: \det_1(\mathcal{I} + \mathcal{K}).$$

- If $\mathcal{K} \in \mathcal{B}_1(\mathcal{H})$, then $\det_2(\mathcal{I} + \mathcal{K}) = e^{-\text{Tr}(\mathcal{K})} \det_1(\mathcal{I} + \mathcal{K})$.

Birman-Schwinger Program for Point Spectrum⁹

- We convert differential operator, $\mathcal{L} - \lambda$ to a Green's operator, $\mathcal{K}(\lambda)$.
- We show that $\mathcal{K}(\lambda)$ is a trace class operator.
- Therefore, $\mathcal{K}(\lambda)$ has a Fredholm determinant and

$$\lambda \in \sigma_{\text{pt}}(\mathcal{L}) \iff \mathcal{I} + \mathcal{K}(\lambda) \text{ is invertible} \iff \det_1(\mathcal{I} + \mathcal{K}(\lambda)) = 0.$$

Pros:

- Don't have to solve stiff systems of differential equations
- For stationary pulses: $E(\lambda) = \det_1(\mathcal{I} + \mathcal{K}(\lambda))$

Cons:

- Fredholm determinant is of an infinite-dimensional operator

BUT

Bornemann devised a method to compute $\det_1(\mathcal{I} + \mathcal{K})$.

⁹Gesztesy, Latushkin, & Makarov, *Arch. Rational Mech. Anal.*, 2007

Conversion to First-Order System

To calculate the spectrum of $\mathcal{L}_\infty$, we convert the system $(\mathcal{L}_\infty - \lambda)\mathbf{u} = 0$ to the first-order **unperturbed** problem

$$\partial_x \mathbf{v}_\infty = \mathbf{A}(\lambda) \mathbf{v}_\infty, \quad \mathbf{v}_\infty = [\mathbf{u} \ \mathbf{u}_x]^T,$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{2 \times 2} & \mathbf{0}_{2 \times 2} \\ \mathbf{B}^{-1}(\lambda - \mathbf{N}_0) & \mathbf{0}_{2 \times 2} \end{bmatrix} \in \mathbb{C}^{4 \times 4}.$$

Similarly, we convert $(\mathcal{L} - \lambda)\mathbf{u} = 0$ to the **perturbed** problem

$$\partial_x \mathbf{v} = (\mathbf{A}(\lambda) + \mathbf{R}(x)) \mathbf{v}, \quad \mathbf{v} = [\mathbf{u} \ \mathbf{u}_x]^T,$$

where the perturbation $\mathbf{R}$ is given by

$$\mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{B}^{-1}\mathbf{M} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{B}^{-1}\mathbf{M}|\mathbf{M}|^{-1/2} & \mathbf{0} \end{bmatrix} \begin{bmatrix} |\mathbf{M}|^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{R}_\ell \mathbf{R}_r.$$

The Unperturbed System

Define $\mathcal{L}_A : H^1(\mathbb{R}, \mathbb{C}^k) \rightarrow L^2(\mathbb{R}, \mathbb{C}^k)$ [$k = 4$] by

$$\mathbf{w} = \mathcal{L}_A \mathbf{v} := -\partial_x \mathbf{v} + \mathbf{A}(\lambda) \mathbf{v}.$$

$$\lambda \in \sigma_{\text{ess}}(\mathcal{L}) \iff \mathcal{L}_A(\lambda) \text{ is not invertible.}$$

For $\lambda \notin \sigma_{\text{ess}}(\mathcal{L})$, the Green's operator $\mathcal{G}_A = \mathcal{L}_A^{-1}$ is given by

$$\mathbf{v}(x) = (\mathcal{G}_A \mathbf{w})(x) = \int_{\mathbb{R}} \mathbf{K}_A(x, y) \mathbf{w}(y) dy,$$

where

$$\mathbf{K}_A(x, y) = \Psi(x - y),$$

with

$$\Psi(x) = \begin{cases} -e^{\mathbf{A}x} \mathbf{Q}, & x \geq 0, \\ e^{\mathbf{A}x} (\mathbf{I} - \mathbf{Q}), & x \leq 0, \end{cases}$$

where $\mathbf{Q}$ is the $k \times k$ projection matrix onto the stable eigenspace of $\mathbf{A}$.

The Perturbed System and Birman-Schwinger Trick

$\lambda \in \sigma_{\text{pt}}(\mathcal{L}) \iff \exists \mathbf{v} \in L^2(\mathbb{R}, \mathbb{C}^k)$ which solves the perturbed system

$$\partial_x \mathbf{v} = (\mathbf{A}(\lambda) + \mathbf{R}(x))\mathbf{v}$$

or

$$[\mathcal{L}_A + \mathbf{R}(x)] \mathbf{v} = 0$$

Birman-Schwinger Trick:

$$\mathcal{L}_A + \mathbf{R} = \mathcal{L}_A[\mathbf{I} + \mathcal{L}_A^{-1}\mathbf{R}] = \mathcal{L}_A[\mathbf{I} + \mathcal{G}_A\mathbf{R}_\ell\mathbf{R}_r],$$

So, for $\lambda \notin \sigma_{\text{ess}}(\mathcal{L})$,

$$\lambda \in \sigma_{\text{pt}}(\mathcal{L}) \iff \mathcal{L}_A + \mathbf{R} \text{ is not invertible}$$

$$\iff \mathbf{I} + \mathcal{G}_A(\lambda)\mathbf{R}_\ell\mathbf{R}_r \text{ is not invertible}$$

$$\iff \mathcal{I} + \mathcal{K}(\lambda) := \mathbf{I} + \mathbf{R}_r\mathcal{G}_A(\lambda)\mathbf{R}_\ell \text{ is not invertible}$$

Explicit Formula for the Green's Kernel

The operator $\mathcal{K}$ is given by

$$\mathcal{K}\phi(x) = \int_{\mathbb{R}} \mathbf{K}(x, y)\phi(y)dy.$$

with kernel

$$\mathbf{K}(x, y) = \begin{cases} -\mathbf{R}_r(x)\mathbf{Q}e^{\mathbf{A}(x-y)}\mathbf{Q}\mathbf{R}_\ell(y), & x \geq y, \\ \mathbf{R}_r(x)(\mathbf{I} - \mathbf{Q})e^{\mathbf{A}(x-y)}(\mathbf{I} - \mathbf{Q})\mathbf{R}_\ell(y), & x < y, \end{cases}$$

Theorem

- $\mathcal{K}$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}, \mathbb{C}^k)$
- $\mathbf{K} \in L^2(\mathbb{R}, \mathbb{C}^{k \times k}) \cap C^0(\mathbb{R}, \mathbb{C}^{k \times k})$.

Proof.

$\mathbf{K}$ is continuous, as $\mathbf{R}_r\mathbf{R}_\ell = \mathbf{0}$. □

General Criteria for being Trace Class¹²

Theorem (ZGL)

Let $\mathcal{K} \in \mathcal{B}_2(L^2(\mathbb{R}, \mathbb{C}^k))$ be given in terms of an L^2 -kernel $\mathbf{K}$ such that

$$(\mathcal{K}\phi)(x) = \int_{\mathbb{R}} \mathbf{K}(x, y)\phi(y)dy,$$

where $\mathbf{K}(x, y)$ is Lipschitz-continuous and $\mathbf{K}$ and its first partial derivatives decay exponentially almost everywhere. Then $\mathcal{K} \in \mathcal{B}_1(L^2(\mathbb{R}, \mathbb{C}^k))$.

- Fredholm¹⁰ and Weidmann¹¹ showed this for scalar kernels on $[-1, 1]$.
- We generalize this to matrix-valued kernels on $\mathbb{R}$.
- And to operator-valued kernels, $\mathbf{K} \in C^{0,1}([-1, 1] \times [-1, 1], \mathcal{B}_2(H))$.

¹⁰Fredholm, *Acta Mathematica*, 1903

¹¹Weidmann, *Mathematische Annalen*, 1966

¹²Zweck, Gallo, Latushkin, *Bol. Soc. Mat. Mex.*, 2025

Lipschitz Continuity of the CQ-CGLE Kernel

Theorem

Suppose that $\psi \in C^1(\mathbb{R}, \mathbb{R}^2)$, ψ, ψ_x decay exponentially, $|\psi| > 0$, on $\mathbb{R}$ and $\epsilon, \gamma \geq 0$. Suppose

$\mathbf{a} = [\epsilon \ \gamma]^T$ and $\mathbf{b} = [\mu \ \nu]^T$ are not both $\mathbf{0}$, and

if $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then $\max_{x \in \mathbb{R}} |\psi(x)|^2 < r$ for certain $r = r(\mathbf{a}, \mathbf{b}) \leq \infty$.

Then $\mathbf{K}$ is Lipschitz continuous.

Proof: By Cauchy's Integral Formula

$$|\mathbf{M}(x)| = \sqrt{(\mathbf{M}^*(x)\mathbf{M}(x))} = \frac{1}{2\pi i} \int_{\gamma} \sqrt{z}(\mathbf{M}^*(x)\mathbf{M}(x) - z)^{-1} dz.$$

- Assumptions imply that $\det(\mathbf{M}(x)) \neq 0$ and so $x \mapsto |\mathbf{M}(x)|^{\pm 1/2}$ is C^1 .
- So, $\mathbf{K}(x, y) \in \mathcal{C}^{0,1}(\mathbb{R}, \mathbb{C}^{k \times k})$, using FTC across diagonal.

Analytical Formula for Fredholm Determinant

Theorem (GZL)

Let $\mathcal{K} \in \mathcal{B}_1(L^2(\mathbb{R}, \mathbb{C}^k))$ have continuous exponentially-decaying kernel. Let

$$K_{\mathbf{j}}^{(n)}(x_1, \dots, x_n) := \det [\mathbf{K}_{j_\alpha j_\beta}(x_\alpha, x_\beta)]_{\alpha, \beta=1}^n,$$

where

$$\mathbf{j} \in J_k^{(n)} = \{(j_1 \dots j_n) : 1 \leq j_\ell \leq k \ \forall \ell\}.$$

Then

$$\det_1(\mathcal{I} + \mathcal{K}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathbf{j} \in J_k^{(n)}} \int_{\mathbb{R}^n} K_{\mathbf{j}}^{(n)}(\mathbf{x}) d\mathbf{x}.$$

Calculating $\det_1(\mathcal{I} + \mathcal{K})$ seems hopeless!?

Approximating the Determinant¹³

To approximate

$$\det_1(\mathcal{I} + \mathcal{K}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathbf{j} \in J_k^{(n)}} \int_{\mathbb{R}^n} K_{\mathbf{j}}^{(n)}(\mathbf{x}) d\mathbf{x},$$

- **Truncate** $\mathbb{R}$ to $[-L, L]$.
- Apply a **quadrature rule** to integrate over $[-L, L]^n$.
- $|\text{Total Error}| \leq |\text{Truncation Error}| + |\text{Quadrature Error}|$

**This approximation of the operator determinant
is given by matrix determinant.**

¹³[Bornemann, 2008]

Approximation by Matrix Determinant

Let $Q = Q_M$ be a quadrature rule, evaluated at M nodes $-L \leq x_1 < x_2 < \dots < x_M \leq L$ and using positive weights $w_1, \dots, w_M$.

Define $\mathbf{K}_Q \in \mathbb{C}^{kM \times kM}$ to be the $M \times M$ block matrix

$$\mathbf{K}_Q = \begin{bmatrix} w_1 \mathbf{K}(x_1, x_1) & w_2 \mathbf{K}(x_1, x_2) & \dots & w_M \mathbf{K}(x_1, x_M) \\ w_1 \mathbf{K}(x_2, x_1) & w_2 \mathbf{K}(x_2, x_2) & \dots & w_M \mathbf{K}(x_2, x_M) \\ \vdots & & & \vdots \\ w_1 \mathbf{K}(x_M, x_1) & w_2 \mathbf{K}(x_M, x_2) & \dots & w_M \mathbf{K}(x_M, x_M) \end{bmatrix},$$

where $\mathbf{K}(x, y)$ is the $k \times k$ submatrix given by the kernel.

Then the discrete determinants are given by

$$d_{Q,1} := \det[\mathbf{I}_{kM \times kM} + \mathbf{K}_Q]$$

$$d_{Q,2} := e^{-\text{Tr}(\mathbf{K}_Q)} \det[\mathbf{I}_{kM \times kM} + \mathbf{K}_Q]$$

Theorem

Suppose that $\mathcal{K} \in \mathcal{B}_1(L^2(\mathbb{R}, \mathbb{C}^k))$ has a kernel whose elements satisfy

- ① $K_{ij} \in \text{Lip}([-L, L] \times [-L, L], \mathbb{C})$.
- ② $\exists C, a > 0$ such that $|K_{ij}(x, y)| \leq Ce^{-a(|x|+|y|)} \quad \forall x, y \in \mathbb{R}$.

Then ($p = 1, 2$)

$$|\det_p(\mathcal{I} + \mathcal{K}) - \det_p(\mathcal{I} + z\mathcal{K}|_{[-L, L]})| \leq C_1 e^{-aL}.$$

Truncation Error decreases exponentially as $L \rightarrow \infty$.

Quadrature Error (Composite Simpson's)¹⁴

Theorem

Let $\mathcal{K} \in \mathcal{B}_p(L^2([-L, L], \mathbb{C}^k))$ have Lipschitz kernel, $\mathbf{K}$. Let Q denote the *composite Simpson's* quadrature rule with spacing Δx . Then

$$|d_{Q,p} - \det_p(\mathcal{I} + \mathcal{K})| \leq C_2 \Delta x.$$

For many CQ-CGLE pulses $\mathbf{K}$ is Lipschitz,

and so $\det_p(\mathcal{I} + \mathcal{K})$ can be calculated with

$$|\text{Total Error}| \leq C_1 e^{-aL} + C_2 \Delta x.$$

¹⁴Result extends to *adaptive* composite Simpson's

Truncation and Quadrature Error

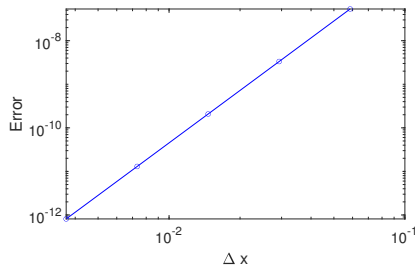
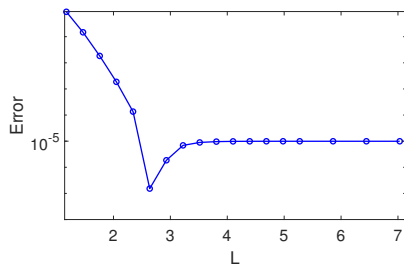


Figure: Truncation Error (Left), Quadrature Error of $O(\Delta x^4)$ (Right)

The Evans Function and the Fredholm Determinant

The kernel of the sech solution of NLSE is trace class. Therefore,¹⁵

$$\det_1(\mathcal{I} + \mathcal{K}(\lambda)) = E(\lambda),$$

where the Evans function is given by¹⁶

$$E(\lambda) = \frac{16\lambda^4}{(1 + \sigma_+)^4(1 + \sigma_-)^4},$$

with $\sigma_{\pm} = \sqrt{1 \pm 2i\lambda}$.

**We validate our approximation of $\det_1(\mathcal{I} + \mathcal{K})$
by comparison with this Evans function.**

¹⁵Gesztesy, Latushkin, & Makarov, *Arch. Rational Mech. Anal.*, 2007

¹⁶See Kapitula and Promislow

Sech Pulse: Evans vs. Fredholm Det

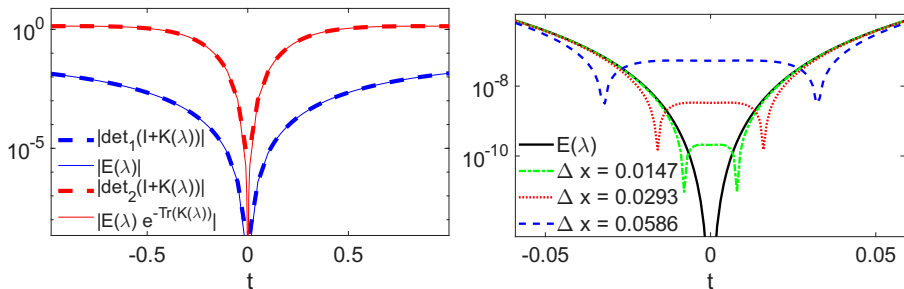


Figure: **Behaviour near order four zero at $\lambda = 0$.** **Left:** Slice along real axis. **Right:** Zoomed in version of results on the left with finer discretizations, Δx , across the pulse.

Validation Results for Sech and CGLE

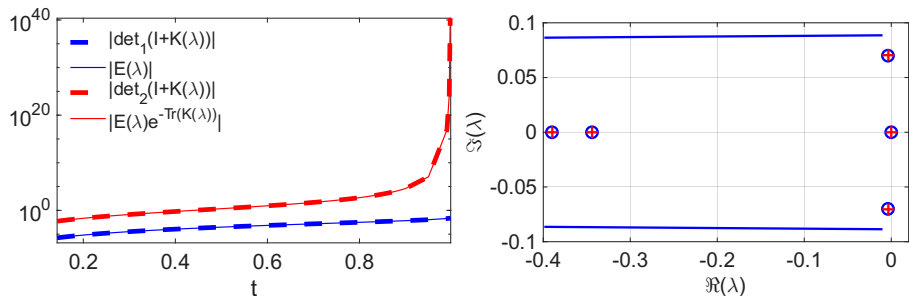


Figure: **Left:** Results for **sech pulse** as λ approaches the edge of $\sigma_{\text{ess}}(\mathcal{L})$ along imaginary axis. **Right:** Spectrum of a **stationary solution of CGLE**: Essential spectrum (solid blue lines) and eigenvalues using $\det_1(\mathcal{I} + \mathcal{K}(\lambda))$ (red pluses) and method of Shen et al. (blue circles).

Zeros of analytic function $\det_1(\mathcal{I} + \mathcal{K}(\lambda))$ were computed via Lyness' method: Contour integrals + Newton.

CGLE: Behaviour near edge of $\sigma_{\text{ess}}(\mathcal{L})$

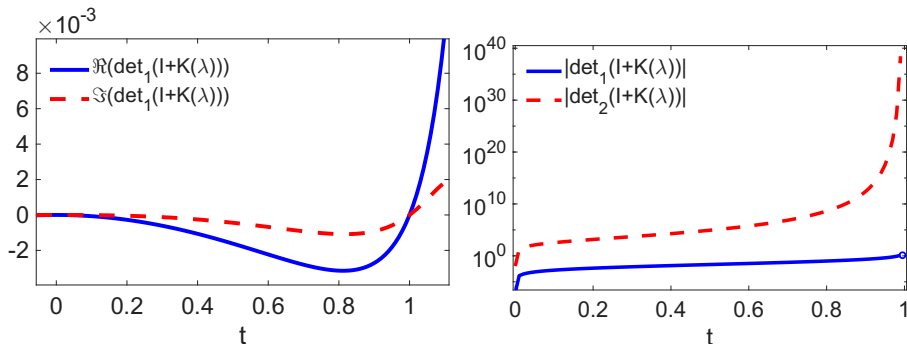


Figure: **Left:** Slice along line $\lambda(t) = \lambda_c t$, between eigenvalues at $\lambda = 0$ and $\lambda_c \approx -0.0033 + 0.0704i$. **Right:** Slice along line joining λ_c to λ_e .

$$\det_2(\mathcal{I} + \mathcal{K}(\lambda)) = e^{-\text{Tr}(\mathcal{K}(\lambda))} \det_1(\mathcal{I} + \mathcal{K}(\lambda)).$$

$$\text{and } \text{Tr}(\mathcal{K}(\lambda)) = \frac{a_+}{\sqrt{\lambda - \lambda_{e,+}}} + \frac{a_-}{\sqrt{\lambda - \lambda_{e,-}}} \rightarrow -\infty \text{ as } \lambda \rightarrow \lambda_{e,\pm}.$$