A Menagerie of Mathematical Models
Active Learning Project #5
Parametrized Surfaces

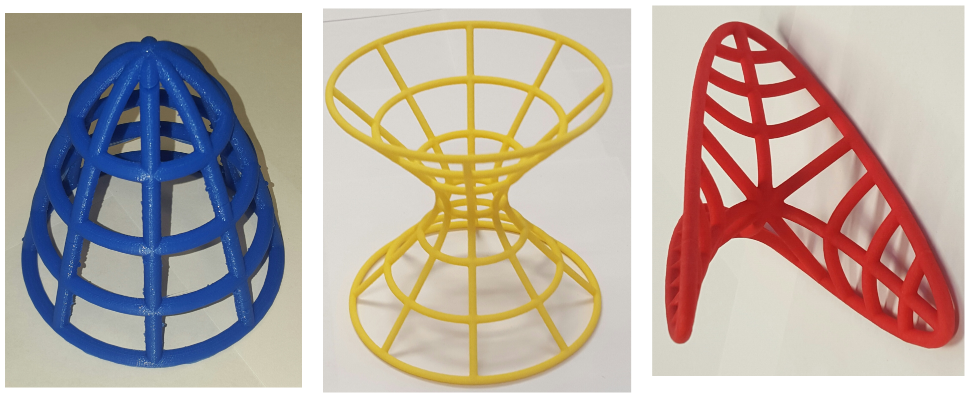

Figure 1: **Left:** Circular paraboloid with polar parameterization. **Middle:** Hyperboloid of 1 sheet with polar parameterization. **Right:** Saddle surface with level curve parameterization. **Rules:** *We want these models to last for many generations of students. Since some of them are delicate please handle with care. Try not to transfer any white board marker ink on your hands to the models. Do not attempt to flex the models. When you have completed the activity put the model back in the box you got it from.*

Assessment: If this project is being assessed, **do Part 2 and one of Parts 1 and 3**. Your small group needs to show the Teaching Assistant (TA) your answers to the questions labeled **TACheck**. Total Points: 10.

We have already seen three methods for describing a plane in space. We can use the same three methods to describe an arbitrary surface in space.

Method	Plane	Sphere	General Surface
Level Surface	$ax + by + cz + d = 0$	$x^2 + y^2 + z^2 - 1 = 0$	$F(x, y, z) = 0$
Graph of Function	$z = Ax + By + C$	$z = \pm\sqrt{1 - x^2 - y^2}$	z=f(x,y)
Surface Parametrization	$\mathbf{x}(s, t) =$ $\mathbf{p} + s\mathbf{u} + t\mathbf{v}$	$\mathbf{x}(\theta, \phi) =$ $(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$	$\mathbf{x}(s, t) =$ $(x(s, t), y(s, t), z(s, t))$

A surface parametrization is a mapping $\mathbf{x} : D \to \mathbb{R}^3$ where D is a domain in $\mathbb{R}^2$. For example, D could be a rectangle or a disk. In coordinates we write $\mathbf{x}(s, t) = (x(s, t), y(s, t), z(s, t))$, for some functions, x, y, z of s and t. Each of the models in Figure 1 was designed using surface parametrizations. The purpose of this project is to learn about surface parametrizations with the aid of these models. The **image** of $\mathbf{x}$ is a two-dimensional surface in $\mathbb{R}^3$. The **parameters**, (s, t), give us a **system of coordinates** on the surface. Suppose you are an ant that lives on a surface. If you and your friends know the formula for a parametrization of this surface, then you can tell your friends where you are on the surface by texting them your coordinates (s, t).

1 Circular paraboloid with polar parameterization

This model was designed using the parametrization

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)) = (\sqrt{v}\cos u, \sqrt{v}\sin u, v), \qquad \text{for } 0 \leq u \leq 2\pi\ ,\ 0 \leq v \leq 4. \quad (1)$$

We can also write this as

$$x = \sqrt{v}\cos u \quad (2)$$

$$y = \sqrt{v}\sin u \quad (3)$$

$$z = v \quad (4)$$

1. **TACheck [1pt]: Equation.** What level surface equation does this surface parametrization correspond to? To work this out, eliminate u and v from Equations (2)–(4) to obtain a single equation relating x, y, and z. Do you recognize this equation?

2. **TACheck [2pts]: Values of u_0, v_0.** Using the parametrization we can construct two families of **grid curves** on the surface. Each grid curve is obtained by setting one of the parameters to be a constant. For example, the $v = 2$ grid curve is the curve on the surface with parametrization

   $$\alpha(u) = (x(u), y(u), z(u)) = (\sqrt{2}\cos u, \sqrt{2}\sin u, 2), \qquad \text{for } 0 \leq u \leq 2\pi. \quad (5)$$

   Similarly, the $u = \frac{\pi}{4}$ grid curve is the curve on the surface with parametrization

   $$\beta(v) = (x(v), y(v), z(v)) = (\sqrt{v}\cos\tfrac{\pi}{4}, \sqrt{v}\sin\tfrac{\pi}{4}, v), \qquad \text{for } 0 \leq v \leq 4. \quad (6)$$

   Instead of fixing v to be $v = 2$ in (5), we could fix it to be $v = v_0$. Different values of v_0 give us different grid curves on the surface. Similarly, different values of u_0 give us different grid curves with $u = u_0$ on the surface.

   The 3D-printed model is an example of a **meshed surface**, since it was constructed using a mesh consisting of a family of $u = u_0$ grid curves and a family of $v = v_0$ grid curves. Identify these two families of curves on the surface. What are the values of u_0 and v_0 that were used to construct this surface mesh?

3. **TACheck [1pt]: Explain.** The domain of our parametrization is a rectangle in (u, v)-space, $\mathbb{R}^2$. This rectangle is $D = [0, 2\pi] \times [0, 4]$. Draw a regular grid with grid spacings $\Delta u = \frac{\pi}{4}$ and $\Delta v = 1$ on this rectangle. Explain why the image of this grid in $\mathbb{R}^3$ is the meshed surface depicted in the model.

4. **TACheck [1pt]: Answer.** The regular grid on D in Question 3 divides D into small rectangles all of which have the **same** area $\Delta A = \Delta u \Delta v$. The image of each of these rectangles is a curved quadrilateral region on the surface. What do you notice about the areas of these quadrilateral regions?[1]

[1]Understanding the answer to this question is important because it is related to understanding how to use surface parametrizations to compute the area of a surface and to integrate functions over surfaces. See the end of the course!

2 Hyperboloid of 1 sheet with polar parameterization

TACheck [5pts]: Parametrization. The level set equation of this surface is

$$x^2 + y^2 - z^2 = 1. \tag{7}$$

Work out the formula for the parametrization used to design the 3D-printed surface. First, use the grid curves on the meshed surface to help identify the two parameters. Then work out a formula for the parametrization. **Hint: You will find it helpful to use cylindrical coordinates.**

3 Saddle surface with level curve parameterization

This surface is the graph of the function

$$z = f(x, y) = x^2 - y^2. \tag{8}$$

The purpose of this activity is to understand how this model was designed.

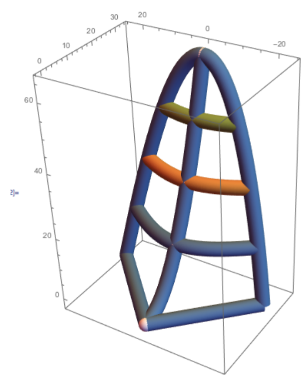

Figure 2: A quarter of the saddle surface.

1. The surface was constructed using four copies of the quarter-saddle meshed surface shown in Fig. 2. Explain how this was done.
2. The meshed saddle surface consists of four types of curves: lines, parabolas, hyperbolae, and one additional curve. Identify each of these types of these curves in the model.
3. **TACheck [1pt]: Parametrizations of Lines.** Write down parametrizations for the two lines in the model.
4. **TACheck [1pt]: Parametrization.** The model only shows that portion of the saddle surface that is inside the cylinder $x^2 + y^2 = 1$. The intersection of the saddle surface and this cylinder is a curve. Identify this curve on the model. Work out a parametrization for it.
5. **TACheck [1pt]: Equation.** The quarter-saddle shown in Fig. 2 is parametrized by

$$x = \sqrt{u} \cosh v \tag{9}$$
$$y = \sqrt{u} \sinh v \tag{10}$$
$$z = u, \tag{11}$$

where $u > 0$. Eliminate the parameters u and v to obtain an equation relating x, y, and z. What does this calculation tell you? **Hint:** Recall that the hyperbolic functions are defined by $\cosh v = \frac{1}{2}(e^v + e^{-v})$ and $\sinh v = \frac{1}{2}(e^v - e^{-v})$. Using these definitions and some algebra you can derive the identity $\cosh^2 v - \sinh^2 v = 1$.

6. **TACheck [1pt]: Parameter and value.** The parabola in the quarter-saddle surface is a grid curve of the parametrization in 5. Which parameter, u or v, is constant on this grid curve and what is its value?

7. **TACheck [1pt]: Parameter and values.** The one-armed hyperbolae in the quarter-saddle surface are grid curves of the parametrization in 5. Which parameter, u or v, is constant on these grid curves and what are its values?