A Menagerie of Mathematical Models
Active Learning Project #2
Saddle Surfaces

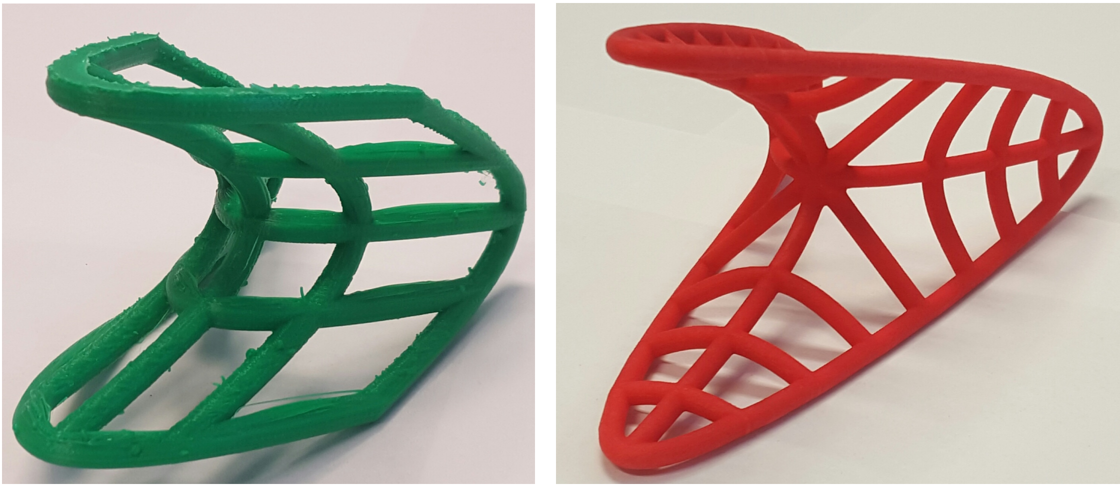

Figure 1: Left: **Green** Saddle Surface. [**Warning:** Some of the so-called green models are actually white, blue, or purple.] Right: **Red** Saddle Surface For this worksheet your group will need one blue model and one red model. **Rules:** *We want these models to last for many generations of students. Since some of them are delicate please handle with care. Try not to transfer any white board marker ink on your hands to the models. Do not attempt to flex the models. When you have completed the activity put the model back in the box you got it from.*

Assessment: If this project is being assessed, your small group needs to show the Teaching Assistant (TA) your answers to the questions labeled **TACheck**. Total Points: 10.

The purpose of this activity is to investigate the geometric structure of the surface given by the graph of the function

$$z = f(x, y) = x^2 - y^2.$$

The mathematical name for this surface is a **hyperbolic paraboloid**. The common name is a **saddle surface**. In reality the surface is an infinitely thin sheet. However, we represent the surface using a **mesh**. You can think of a **surface mesh** as being like a curved fishing net that is draped over the surface. You could imagine making such a fishing next by laying a collection of ropes on the surface, some going one way on the surface, some going another way. Wherever two ropes cross you tie them together. With the green and red 3D models we have used thin plastic tubes instead of ropes, but the idea is the same.

1. **TACheck [1pt]: Model position.** For each model identify where the origin is and which direction the three coordinate axes go. Draw x and y coordinate axes on a sheet of paper, put the paper on the table, and position the model on top of the paper. (The z-axis would of course be perpendicular to the paper.)

2. The shadow of the green model on the xy-plane is a rectangular grid that covers the rectangle with $-2 \le x \le 2$ and $-2 \le y \le 2$. Use the flashlight app on your phone to cast a shadow

of the green model onto the table. You should be able to see a slightly distorted version of this grid. *Whenever you use a flashlight to cast a shadow in these projects you need to have the* **flashlight at least two feet above the model** *so that the rays of light are falling almost vertically down onto the model.*

3. **TACheck [2pts]: Equations.** The green model is made using **two families of curves**. Each of these curves can be obtaining by slicing the surface in a plane. Textbooks often call these curves "traces", but we will use the more descriptive term "slices".
   (a) The slice of the surface in the plane $y = k$ is the curve $z = x^2 - k^2$. In the model we sliced at $k = 0, \pm 1, \pm 2$. Identify these curves on the model.
   (b) The slice of the surface in the plane $x = k$ is the curve $z = k^2 - y^2$. In the model we sliced at $k = 0, \pm 1, \pm 2$. Identify these curves on the model.
4. **TACheck [2pts]: Explanation.** By looking at the green model, explain how the curves in part 3 can all be obtained by taking a single curve and moving it around in space.[1]
5. **TACheck [2pts]: Equations.** Next let's turn our attention to the red model.
   (a) Sketch the curves you get when you cast a shadow of the red model on the table.
   (b) Explain how each of the curves in the red model can be obtaining by slicing the surface in a plane. (Actually there is one curve in the model that can't be obtained this way. Can you find it?) Write down the equations for these curves and their names. [Hint: The sketch you drew in part 5a may be helpful here.]
6. **TACheck [2pts]: Convince TA.** Do a 45° rotation of the piece of paper on which you drew the xy-coordinate axes, but don't rotate the model. When you do this, the (red) model turns into the graph of the function

$$z = f(x, y) = xy.$$

   Convince yourself of this fact. To do so you can sketch the slices (traces) of $z = xy$ in vertical planes $z = k$ and compare them to the shadows cast on the rotated xy-plane by the red model. In addition, can you work out what vertical plane to slice the red model in to get the upward parabola in the red model? How do you get a downward parabola?
7. **TACheck [1pt]** Reorient one of the models so that it becomes the graph of the function

$$x = f(y, z) = y^2 - z^2.$$

8. Hold the red model so that the vector from the origin of the model to your eye is roughly parallel to the vector $\mathbf{v} = (1, 1, 2)$. Now attempt to make a sketch of the saddle surface. (This is a challenging but very worthwhile exercise. You may wish to attempt it more than once.)
9. **Geometric Imagination (GI) Builder:** [*Do this after class*] Try to visualize the surface that is the graph of the function

$$z = f(x, y) = 4x^2 - y^2.$$

   How does it differ from the saddle surface models? If you had to name this surface, what would you call it?

[1] If you did Project #1: How does this process differ from what you did in Active Learning Project #1 for the circular paraboloid?