A Menagerie of Mathematical Models

Active Learning Project #6
Ruled Surfaces

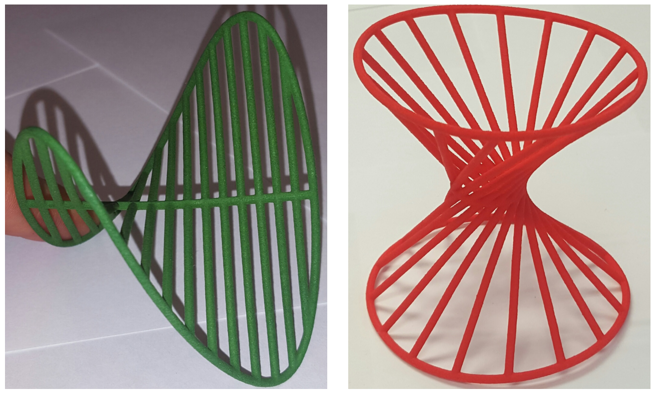

Figure 1: Ruled surfaces: **Left:** Saddle surface; **Right:** Hyperboloid of 1 sheet.

1 Saddle surface as a ruled surface

You will recognize that the green model is a saddle surface. However, it is constructed very differently from how you have seen saddle surfaces constructed so far in this course. The model suggests that you can sweep out a saddle surface using two broomsticks, A and B.

1. Use the model to help you visualize the following scenario. Two men hold the two ends of broomstick A. Two women hold the two ends of broomstick B. The two broomsticks are arranged so that they touch at one point and are at right angles to each other. The women holding the ends of broomstick B walk parallel to broomstick A (one on either side of broomstick A). As they walk, broomstick B slides along broomstick A. In addition, as they go the women rotate broomstick B about broomstick A. The model suggests that if the woman rotate broomstick B at the correct rate, then the surface that broomstick B sweeps out is a saddle surface.

   Put another way, the model suggests that, even though it is a **curved surface**, a saddle surface can be built from **straight lines**.[1] Surfaces with this property are called **ruled surfaces.**

2. Let's use surface parametrizations to work out how the women should rotate broomstick B so as to sweep out a saddle surface. Let's suppose that broomstick A is aligned with the x-axis and that at time t the point of intersection of the two broomsticks is given by

   $$\mathbf{p}(t) = (t, 0, 0). \tag{1}$$

   If we let $\mathbf{v}(t)$ be the unit length tangent vector to broomstick B at time t, then the surface swept out by broomstick B is parametrized by

   $$\mathbf{x}(t, s) = \mathbf{p}(t) + s\mathbf{v}(t). \tag{2}$$

   Explain why this formula holds. Identify the $t = t_0$ grid curves on the 3D printed model.

[1] So you could make a Pringle chip by cutting French fries to the correct lengths and glueing them together!

3. Next, let $\theta(t)$ be the angle that the unit tangent vector $\mathbf{v}(t)$ makes with the vector $\mathbf{j}$. Explain why $\mathbf{v}(t)$ is given by

$$\mathbf{v}(t) = (0, \cos\theta(t), \sin\theta(t)). \tag{3}$$

4. Combine equations (1), (2), and (3) to conclude that the parametrization of the surface swept out by broomstick B is given by

$$x = t \tag{4}$$
$$y = s\cos\theta(t) \tag{5}$$
$$y = s\sin\theta(t). \tag{6}$$

Of course we still don't know how the women should rotate broomstick B so that this surface is a saddle surface, *i.e.*, we don't know how to choose θ as a function of t.

5. Recall from Project #5, that a saddle surface contains two lines in the plane $z = 0$. These lines are the asymptotes of the hyperbolae that are the level curves for the saddle surface. Convince yourself by looking at the model that one of these lines is broomstick A. Where is the other line? Consequently, as you discovered in Project #2, our saddle surface should be the graph of the function

$$z = xy. \tag{7}$$

6. Substitute equations (4), (5), and (6) into (7) to solve for θ in terms of t.

7. Suppose that the men hold broomstick A at waist level and that the women walk at a speed of 1 m/s parallel to broomstick A keeping a distance of 1 m from that broomstick. At time t, how high above/below waist level should the women hold broomstick B? What happens as $t \to \pm\infty$?

2 Hyperboloid of 1 sheet as a ruled surface

The level set equation for the red surface is

$$x^2 + y^2 - z^2 = 1. \tag{8}$$

1. By examining the model, convince yourself that the red model is also a ruled surface.

2. Describe how to slide a broomstick around a hoop to sweep out this surface. **Hint:** Choose the hoop to be the waist of the surface.

3. Parametrize the hoop curve. Call this parameterization $\mathbf{p}(t)$.

4. By looking carefully at the model explain why the tangent vector to the broomstick at time t is of the form $\mathbf{v}(t) = \mathbf{p}'(t) + c\mathbf{k}$, for some scalar c.

5. Using your formulae for $\mathbf{p}$ and $\mathbf{v}$ write down a parametrization for the ruled surface.

6. Plug your formula for the parametrization into the level set equation $x^2 + y^2 - z^2 = 1$ for the surface to determine the constant c.

7. Hopefully, you discovered that there are two possible values for c. This suggests that there is a second way to represent the hyperboloid of 1 sheet as a ruled surface. By looking at the model, can you visualize it?

8. Let a, b and c be positive constants. Is the surface $ax^2 + by^2 - cz^2 = 1$ a ruled surface?