A Menagerie of Mathematical Models

Active Learning Project #8
Hills and Valleys

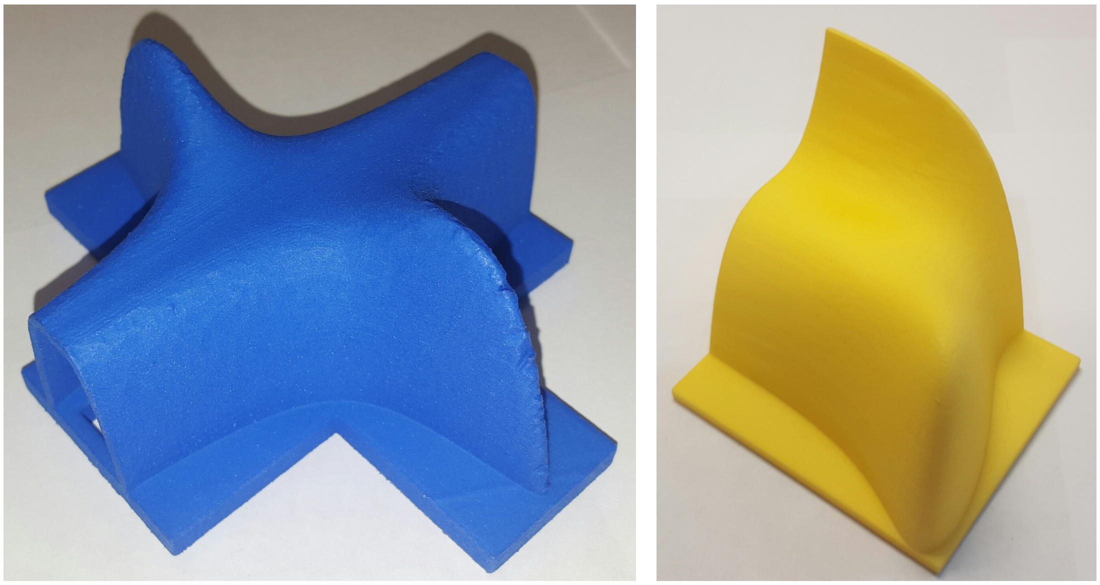

Figure 1: **Left:** Blue model. **Right:** Yellow model.

Assessment: If this project is being assessed, **do one of the two parts**. Your small group needs to show the Teaching Assistant (TA) your answers to the questions labeled **TACheck**. Total Points: 10.

The point of this project is to show that functions, $z = f(x, y)$ of two variables can have properties that functions, $y = f(x)$, of one variable cannot.

1. The blue model is the graph of the function

$$z = f(x, y) = -(x^2 - c^2)^2 - (x^2 y - cx - c)^2, \qquad \text{for some constant } c \neq 0. \tag{1}$$

For the model we chose the constant to be $c = 10$.

(a) **TACheck [1pt].** Use the Chain Rule from Calculus I to show that

$$f_x = -4x(x^2 - c^2) - 2(x^2 y - cx - c)(2xy - c), \tag{2}$$

$$f_y = -2x^2(x^2 y - cx - c). \tag{3}$$

(b) **TACheck [2pts].** Check that the following two points are critical points of f:

$$(x_1, y_1) = (c, 1 + \frac{1}{c}) \qquad \text{and} \qquad (x_2, y_2) = (-c, -1 + \frac{1}{c}). \tag{4}$$

(c) **TACheck [2pts].** By solving the critical point equations, (2) and (3), show that (x_1, y_1) and (x_2, y_2) are the *only* critical points of f.

(d) **TACheck [2pts].** Identify the locations of the two critical points of f on the model and use the model to classify them as local maxima, local minima, or saddle points.

(e) The formulae for the second derivatives of f are a little nasty. So it is a pain to use the Second Derivative Test to classify the critical points of f. However, the function f has some nice properties that enable us to show that the function f attains its *absolute* maximum value at both critical points. Consequently, these critical points must also be *local* maxima of f!

i. **TACheck [1pt].** What is the largest possible value that f could take?

ii. **TACheck [1pt].** Explain why f attains this largest possible value at (x_1, y_1) and at (x_2, y_2).

(f) **TACheck [1pt].** Let $y = f(x)$ be a differentiable function whose domain is the entire real line. Is it possible for f to have exactly two critical points, both of which are local maxima?

2. The yellow model is the graph of the function

$$z = g(x, y) = 3xe^y - x^3 - e^{2y}. \tag{5}$$

(a) **TACheck [1pt].** As best you can, identify the locations of the critical points of g on the model and use the model to classify them as local maxima, local minima, or saddle points.

(b) **TACheck [4pts].** Use multivariable calculus to calculate the critical points of the function g and use the Second Derivative Test to classify them.

(c) **TACheck [1pt].** By examining the model explain why it appears that g does not have an absolute maximum on $\mathbb{R}^2$.

(d) **TACheck [3pts].** Do a calculation to show that g does not have an absolute maximum on $\mathbb{R}^2$. **Hint:** Let $h(y) = g(-e^{-y}, y)$. Work out the formula for h and show that h does not have an absolute maximum on $\mathbb{R}$. Explain why this tells us that g does not have an absolute maximum on $\mathbb{R}^2$. Sketch the graph of $x = -e^{-y}$ in the (x, y)-plane and identify where it is on the model.

(e) **TACheck [1pt].** Let $y = f(x)$ be a differentiable function whose domain is the entire real line. Is it possible for f to have exactly one critical point which is a local maximum but not a global maximum?