A Menagerie of Mathematical Models

Active Learning Project #4
Limits

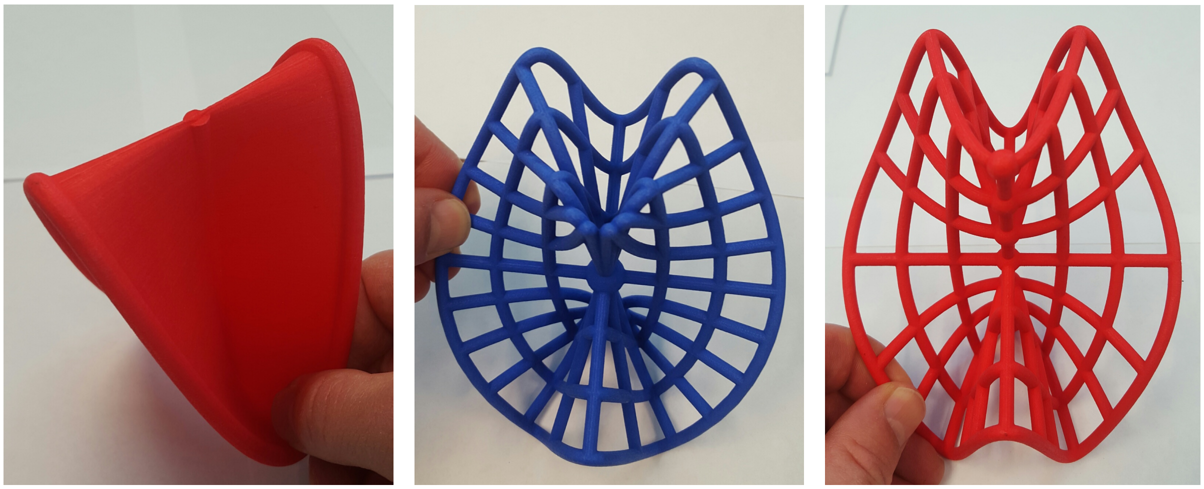

Figure 1: Left: **Red Sheet**, Center: **Blue Mesh**, Right: **Red Mesh**. **Rules:** *We want these models to last for many generations of students. Please handle with care. Try not to transfer any white board marker ink on your hands to the models. When you are finished, put the models back in the box you got them from.*

In the Lectures, we learned that a general strategy for show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist is to find two curves C_1 and C_2 in the (x,y)-plane that go to the origin and have the property that

$$\lim_{(x,y)\to(0,0) \text{ along } C_1} f(x,y) = L_1,$$
$$\lim_{(x,y)\to(0,0) \text{ along } C_2} f(x,y) = L_2$$

but that

$$L_2 \neq L_1.$$

However, we can often gain more geometric insight by considering an entire family of curves each of which goes to the origin, instead of just two curves. For example, we could consider the family of all straight lines $y = kx$ for different values of k.

1 The Red Sheet Model

The purpose of this part of the project is to understand why the limit as $(x,y) \to (0,0)$ of the function

$$f(x,y) = \frac{2xy}{x^2+y^2} \tag{1}$$

does not exist.

1. Explain why the function f is defined for every $(x,y) \in \mathbb{R}^2$ except $(x,y) = (0,0)$.

2. Explain why the Red Sheet model is the graph of a function whose domain is the set of all points in the disc $x^2+y^2 \leq 1$ except the origin. This function is the one given by Equation (1).

3. Calculate $\lim_{(x,y)\to(0,0) \text{ along } y=kx} f(x,y)$.

4. Use the calculation you did in 3 to explain why $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

5. Explain why the calculation you did in 3 shows that over the line $y = kx$ the height of the graph of f is *constant*. What is this constant in terms of k?

6. Now use the model to explain why height of the graph of f is constant over each of the lines $y = kx$.

7. Use the model to explain why $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. In particular, suppose that an ant (Ant A) walks into the origin along a straight line in the (x, y)-plane and that the ant's friend (Ant B) walks immediately above her on the surface. Convince yourself that the height Ant B ends up at depends on the direction in the plane that Ant A chooses to walk into the origin.

8. Another way to investigate whether exists $\lim_{(x,y)\to(0,0)} f(x,y)$ is to convert to polar coordinates. Show that
$$f(r\cos\theta, r\sin\theta) = \sin(2\theta).$$
Explain why this calculation shows that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

9. Show that the intersection of the graph of f with the cylinder $x^2 + y^2 = 1$ is the curve
$$(x, y, z) = \mathbf{r}(t) = (\cos t, \sin t, \sin(2t)).$$
Identify this curve on the model.

2 The Blue Mesh and Red Mesh Models

The purpose of this part of the project is to understand why the limit as $(x, y) \to (0, 0)$ of the function
$$g(x,y) = \frac{2x^2y}{x^4 + y^2} \tag{2}$$
does not exist.

1. Explain why the function g is defined for every $(x, y) \in \mathbb{R}^2$ except $(x, y) = (0, 0)$.

2. Explain why the Red Mesh and Blue Mesh models depict the graph of a function whose domain is the set of all points in the disc $x^2 + y^2 \leq 1$ except the origin.

3. Although the two meshes are different, by comparing the overall shape of the two surfaces convince yourself that the Red Mesh and Blue Mesh surfaces are both the graph of the *same* function. This function is the one given by Equation (2).

4. Show that for every value of k, we have that

$$\lim_{(x,y)\to(0,0) \text{ along } y=kx} g(x,y) = 0. \tag{3}$$

5. Use the Blue Mesh model to verify that Equation (3) is indeed true. **Hint:** Use the flashlight app on your phone to look at the shadow of the mesh on the table. Can you identify lines $y = kx$ for different values of k in the shadow? Where do they come from on the surface?

6. What can you conclude from Equation (3) about $\lim_{(x,y)\to(0,0)} g(x,y)$?

7. **Before proceeding further check with your TA that you have answered Question 6 correctly!**

8. Another family of curves that goes into the origin is the family of parabolas $y = kx^2$ for different values of k. Calculate

$$\lim_{(x,y)\to(0,0) \text{ along } y=kx^2} g(x,y). \tag{4}$$

9. Use the calculation you did in 8 to explain why $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist.

10. Explain why the calculation you did in 8 shows that over the parabola $y = kx^2$ the height of the graph of g is *constant*. What is this constant in terms of k?

11. Now use the Red Mesh model to explain why height of the graph of g is constant over each of the parabolas $y = kx^2$.

12. Finally, use the Red Mesh model to explain why $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist. Similarly to Question 7 in Section 1 above, explain what's going on here using Ant A and Ant B.

13. What is the moral of the story of the Blue Mesh and Red Mesh models?

14. **Geometric Imagination (GI) Builder:** [*Do this after class*] Can you visualize the graph of the function

$$h(x,y) = \frac{2x^3y}{x^6 + y^2} \ ? \tag{5}$$