A Menagerie of Mathematical Models

Lecture Demo #2
From the Temple of Viviani to MH 370

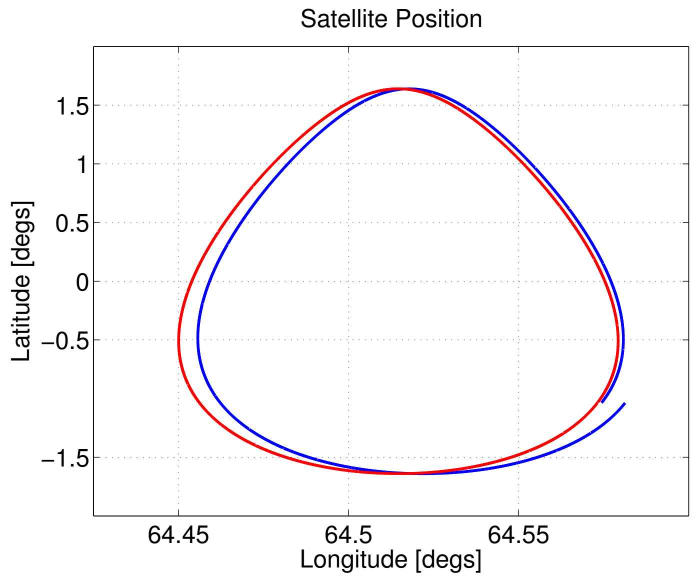

Figure 1: **Left:** The Temple of Viviani; **Right:** The projection onto the earth of the orbit of the Inmarsat 3-F1 satellite on the day that Malaysia Airlines flight 370 disappeared.

Legend has it that the Temple of Viviani was an ancient Greek temple constructed by intersecting a sphere with an off-center cylinder. The walls of the temple are formed by the cylinder and the roof by the sphere. Of course, the floor of the temple is in the equatorial plane of the sphere, and the temple is only the top half of the intersection of the cylinder and the sphere. For the yellow model, the equation of the sphere is

$$x^2 + y^2 + z^2 = 4$$

and the equation of the off-center cylinder is

$$(x - 1)^2 + y^2 = 1.$$

Consequently the sphere and cylinder intersect in the point $(2, 0, 0)$. This surface was extensively studied by mathematicians from about the 1600's onwards, and is even discussed in an early 20th Century book on Differential Geometry by Professor Struik. Many people tried to visualize the shape of the curve of intersection of the two surfaces but apparently few if any got it exactly correct. Thomas Banchoff at Brown used computer graphics to visualize it. Now I have made a 3D-printed model of this surface.

The Temple of Viviani is closely related to the orbit of the satellite that was used to reconstruct the flight path of Malaysia airlines flight 370.

In 2014 I got very interested in the fate of MH 370, which disappeared in the Southern Indian Ocean. A team of satellite engineers used a sequence of automatic communications signals between the MH 370 airplane and the Inmarsat 3-F1 satellite to reconstruct the flight path of the satellite. Key to this research was knowledge of the orbit of the satellite, whose position was known from measured data. The actual path of the satellite over the surface of the earth is shown in the blue curve on the right on the Figure.

This orbit is approximately a geosynchronous orbit. A satellite in a geosynchronous orbit returns to the same position in the sky once a day. Note that the orbit is not geostationary, in that it is not located in the same position in the sky all the time. The fact that the orbit was geosynchronous rather than geostationary was of vital importance for the reconstruction of MH 370 since with a geostationary orbit it would be impossible to tell whether the airplane had travelled into the northern rather than the southern hemisphere. However, the geosynchronous nature of the satellite orbit breaks a symmetry in the problem which enabled the satellite engineers to definitely conclude that the airplane went south. Debris recovered from MH 370 near Madagascar many months after the plane went down confirms this conclusion.

I developed a mathematical model of the satellite orbit, which is described in my SIAM Review article. Briefly, the satellite orbit is in a plane that is slightly tilted with respect to the equator. Furthermore, it rotates about the axis orthogonal to this plane at the same angular velocity as the earth rotates about its axis. This fact accounts for the fact that the satellite returns to the same point in the sky once a day. If the distance of the satellite from the earth were a constant, that is, if the orbit was a perfect circle, then the projection of the satellite onto the earth would be a curve called an analema. The connection with the Temple of Viviani is that an analema is a curve on a sphere that is the intersection of the sphere with an off-center cylinder. However in the case of a geosynchronous orbit the cylinder is a very thin one those axis is located very close to the equator of the sphere. That is, the equation of the cylinder is of the form

$$[x - (2 - \epsilon)]^2 + y^2 = \epsilon^2,$$

for some small positive number ϵ. In this case the analema then looks a bit like a tear-drop shaped ear-ring. In reality, the orbit of the Inmarsat 3-F1 satellite was not a perfect circle. Although it orbits in a plane, the radius of the orbit oscillates slightly due to the tidal gravitational effects of the moon. As a consequence the projection of the satellite orbit onto the earth is not actually an analema.