A Menagerie of Mathematical Models

Lecture Demo #1
Humpty Dumpty on a Horse

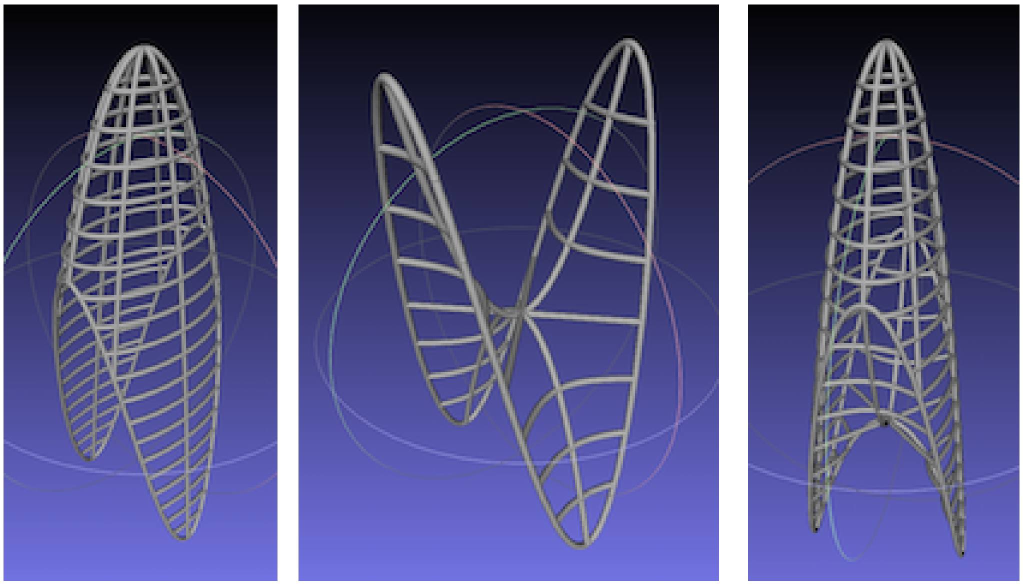

Figure 1: **Left:** Humpty Dumpty; **Center:** The Horses' Saddle; **Right:** Humpty Dumpty on a Horse.

These models can be used to visualize the volume between an elliptical paraboloid and a saddle surface. The elliptical paraboloid is the graph of the function

$$z \;=\; f(x,y) \;=\; 2 - x^2 - 3y^2$$

and the saddle surface is the graph of the function

$$z \;=\; g(x,y) \;=\; x^2 - y^2.$$

The volume can be filled with vertical French fries. The French fries are labelled by their position (x,y) in the xy-plane. The French fry at (x,y) has the range of z-values given by

$$g(x,y) \;\le\; z \;\le\; f(x,y).$$

The two surfaces meet on the curve in space consisting of all points (x,y,z) so that

$$g(x,y) \;=\; z \;=\; f(x,y).$$

If we convert the equation $g(x,y) \;=\; f(x,y)$ to polar coordinates we obtain the equation $r = 1$. Consequently, the curve lies over the circle $r = 1$ in the xy-plane. Therefore, this curve is the intersection of a cylinder and a saddle surface. So it is parametrized by

$$(x,y,z) \;=\; \mathbf{r}(t) \;=\; (\cos t, \sin t, \cos^2 t - \sin^2 t).$$

The volume is then given

$$V \;=\; \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left[f(r\cos\theta, r\sin\theta) - g(r\cos\theta, r\sin\theta)\right]\; r\,dr\,d\theta.$$